Vector Calculus Lecture 26

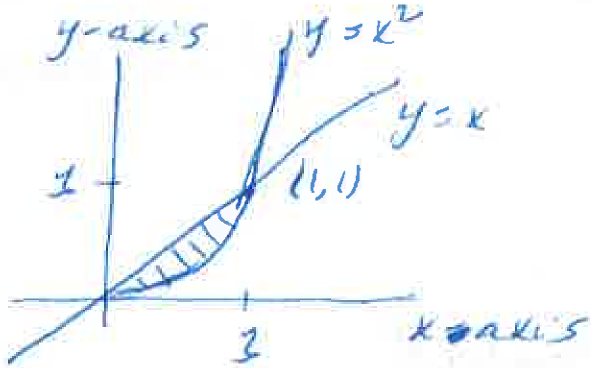
21.09.2018 ①
UniHel6
R. Ram

§5.1 Example 1 Let $\vec{F}(x,y) = (xy^2, y+x)$.

Verify Green's theorem for the region bounded by

$y = x^2$, $y = x$ with $y \geq 0$ and $x \geq 0$

Solution



Green's theorem:
$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Since $\vec{F} = (xy^2, y+x)$ then $P = xy^2$
$Q = y + x$.

So $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2xy$ and

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -\int_{x=0}^{x=1} \int_{y=x}^{y=x^2} (1-2xy)\,\cancel{dx\,dy}\,dy\,dx$$

$$= -\int_{x=0}^{x=1} y - xy^2 \Big]_{y=x}^{y=x^2} dx = -\int_{x=0}^{x=1} \left(x^2 - x x^4 - (x - x x^2)\right) dx$$

$$= -\int_{x=0}^{x=1} \left(-x + x^2 + x^3 - x^5\right) dx = -\left(-\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^6}{6}\right)\Big]_{x=0}^{x=1}$$

$$= -\left(-\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - (0+0+0-0)\right)$$

$$= -\left(-\frac{1}{6} + \frac{3}{12} - \frac{1}{6}\right) = -\left(-\frac{4}{12} + \frac{3}{12}\right) = -\left(-\frac{1}{12}\right) = \frac{1}{12}$$

Then

$$\int_{\partial D} P\,dx + Q\,dy = \int_{c_1} P\,dx + Q\,dy + \int_{c_2} P\,dx + Q\,dy$$

where

$$\vec{c}_1(t) = (t, t^2) \quad \text{for} \quad 0 \le t \le 1$$

$$\vec{c}_2(t) = (1-t, 1-t) \quad \text{for} \quad 0 \le t \le 1$$

So

$$\int_{c_1} P\,dx + Q\,dy = \int_{c_1} \left(xy^2 \frac{dx}{dt} + (y+x)\frac{dy}{dt}\right) dt$$

$$= \int_{t=0}^{t=1} \left(t(t^2)^2 \cdot 1 + (t^2+t)2t\right) dt$$

$$= \int_{t=0}^{t=1} (t^5 + 2t^3 + 2t^2)\,dt = \frac{t^6}{6} + \frac{t^4}{2} + \frac{2t^3}{3}\Big]_{t=0}^{t=1}$$

$$= \frac{1}{6} + \frac{1}{2} + \frac{2}{3} - (0+0+0) = \frac{8}{6}$$

$$\int_{c_2} P\,dx + Q\,dy = \int_{c_2} \left(xy^2 \frac{dx}{dt} + (y+x)\frac{dy}{dt}\right) dt$$

$$= \int_{c_2} \left((1-t)(1-t)^2(-1) + (1-t+1-t)(-1)\right) dt$$

$$= \int_{t=0}^{t=1} \left((1-t)^3 + 2(1-t)\right)(-1)\,dt = \frac{(1-t)^4}{4} + (1-t)^2\Big]_{t=0}^{t=1}$$

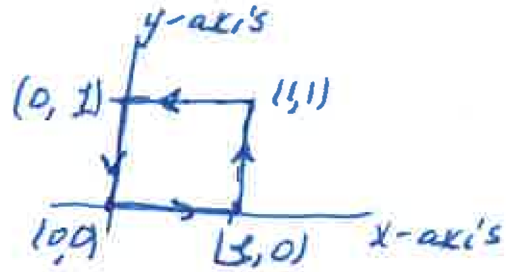
$$= 0 + 0 - \left(\frac{1}{4} + 1\right) = \frac{-5}{4}$$

So

$$\int_{\partial D} P\,dx + Q\,dy = \frac{8}{6} - \frac{5}{4} = \frac{16}{12} - \frac{15}{12} = \frac{1}{12}$$

§5.1 Example 2 Evaluate $\int_C x^2\,dx + xy\,dy$

of



Solution Use Green's theorem

$$\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

So

$$\int_C x^2\,dx + xy\,dy = \iint_D (y - 0)\,dx\,dy$$

$$= \int_{y=0}^{y=1}\int_{x=0}^{x=1} y\,dx\,dy = \int_{y=0}^{y=1} yx\Big]_{x=0}^{x=1} dy = \int_{y=0}^{y=1} y\,dy$$

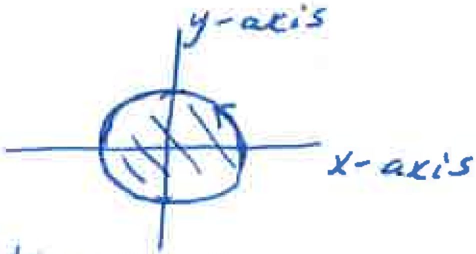
$$= \frac{y^2}{2}\Big]_{y=0}^{y=1} = \frac{1}{2}$$

§5.1 Example 3 Let

$$\vec{F}(x,y) = (2y + e^x, x + \sin(y^2))$$

Evaluate $\int_C \vec{F} \cdot d\vec{s}$ where C is the unit circle centred at $(0,0)$ traversed anticlockwise.

Solution:



Use Green's theorem

$$\int_C \vec{F} \cdot d\vec{s} = \int_{C = \partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

with $P = 2y + e^x$
$Q = x + \sin(y^2)$ so $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 - 2 = -1.$

So

$$\int_C \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_D (-1)\,dx\,dy$$

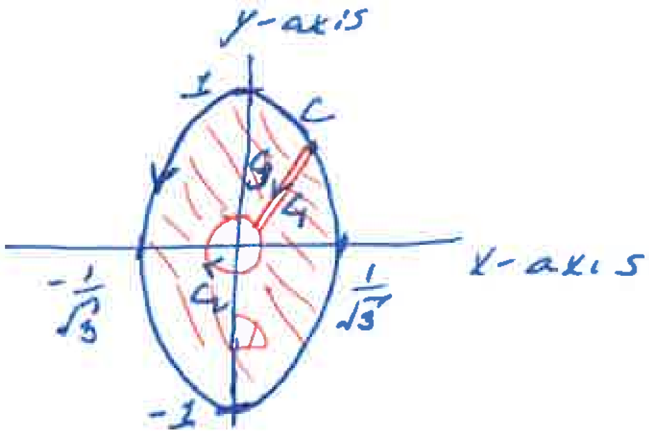
$$= (-1)\left(\begin{matrix}\text{Area of}\\ D\end{matrix}\right) = (-1)\,\pi\,1^2 = -\pi.$$

35.1 Example 4 Evaluate $\int_C \vec{F}\cdot d\vec{s}$ where

$$\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

and C is the curve $3x^2+y^2=1$ oriented anticlockwise.

Solution:



The problem with applying Green's theorem straight up is that is not C^1 at $(x,y)=(0,0)$. So we make a path to avoid $(x,y)=(0,0)$ in the region D. Then

$$\partial D = C \cup C_1 \cup C_2 \cup C_3$$

let $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$ so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-x\,2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \left(\frac{+y\,2y}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}\right)$$

$$= \frac{-2x^2-2y^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} = 0.$$

So

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D 0\,dx\,dy = 0.$$

So
$$0=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\int_{\partial D}P\,dx+Q\,dy=\int_{\partial D}\vec{F}\cdot d\vec{s}$$

$$=\int_C\vec{F}\cdot d\vec{s}+\int_{C_1}\vec{F}\cdot d\vec{s}+\int_{C_2}\vec{F}\cdot d\vec{s}+\int_{C_3}\vec{F}\cdot d\vec{s}$$

Since C_1 and C_3 are the same curve but in opposite directions then

$$\int_{C_1}\vec{F}\cdot d\vec{s}=-\int_{C_3}\vec{F}\cdot d\vec{s}.$$

So
$$0=\int_C\vec{F}\cdot d\vec{s}+\int_{C_2}\vec{F}\cdot d\vec{s}.$$

So $$\int_C\vec{F}\cdot d\vec{s}=-\int_{C_2}\vec{F}\cdot d\vec{s}=\int_{R_2}\vec{F}\cdot d\vec{s}$$

where R_2 is the reverse of C_2.

$$\vec{R}_2=(\rho\cos t,\rho\sin t)\quad\text{for}\quad 0\le t\le 2\pi$$

where ρ is the radius of $\vec{R}_2$.

So
$$\int_{R_2}\vec{F}\cdot d\vec{s}=\int_{R_2}P\,dx+Q\,dy=\int_{R_2}\left(\frac{-y}{x^2+y^2}\frac{dx}{dt}+\frac{x}{x^2+y^2}\frac{dy}{dt}\right)dt$$

$$= \int_{R_2} \left(-\frac{\rho \sin t}{\rho^2} (-\rho \sin t) + \frac{\rho \cos t}{\rho^2} \rho \cos t \right) dt$$

$$= \int_{t=0}^{t=2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_{t=0}^{t=2\pi} dt$$

$$= t \Big]_{t=0}^{t=2\pi} = 2\pi - 0 = 2\pi.$$

So $\int_C \vec{F} \cdot d\vec{s} = \int_{R_2} \vec{F} \cdot d\vec{s} = 2\pi.$