Vector Calculus Lect 27

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Stokes theorem $\int_D dw = \int_{\partial D} w$

Case 1: Stokes theorem

Let $S \subseteq \mathbb{R}^3$ be a surface and $\vec{F} \in \Omega^1_{\mathbb{R}^3}$. Then

$$\iint_S \mathrm{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

Case 2: Divergence theorem

Let $\Omega \subseteq \mathbb{R}^3$ be a volume and $\vec{G} \in \Omega^2_{\mathbb{R}^3}$. Then

$$\iiint_\Omega \mathrm{div}(\vec{G})\, dV = \iint_{\partial\Omega} \vec{G} \cdot d\vec{S}$$

Case 3: Green's theorem

Let $D \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$ be an area and $\vec{F} \in \Omega^1_{\mathbb{R}^2}$. Then

$$\iint_D \mathrm{curl}(\vec{F}) \cdot \hat{k}\, dx\,dy = \int_{\partial D} \vec{F} \cdot d\vec{s}.$$

Case 4: Divergence theorem in $\mathbb{R}^2$

Let $D \subseteq \mathbb{R}^2$ be an area and $\vec{G} \in$. Then

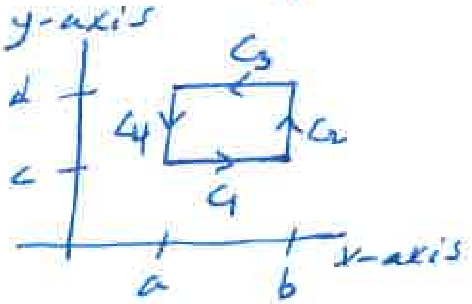
$$\iint_D \mathrm{div}(\vec{G})\, dx\,dy = \int_{\partial D} \vec{G} \cdot \hat{n}\, ds$$

Case 5: Fundamental theorem of calculus.

Let $[a,b] \subseteq \mathbb{R}^1$ and $f \in \Omega^0_{\mathbb{R}^1}$. Then

$$\int_{[a,b]} \frac{df}{dt}\, dt = \int_{\partial[a,b]} f\, ds = f(b) - f(a)$$

§5.1 Green's theorem Let $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ and verify Green's theorem for the region



$c_1(x) = (x,c)$ with x from a to b
$c_2(y) = (b,y)$ with y from c to d
$c_3(x) = (x,d)$ with x from b to a
$c_4(y) = (a,y)$ with y from d to c.

Solution

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \frac{\partial Q}{\partial x}\,dx\,dy - \iint_D \frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_{y=c}^{y=d}\int_{x=a}^{x=b} \frac{\partial Q}{\partial x}\,dx\,dy - \int_{x=a}^{x=b}\int_{y=c}^{y=d} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_{y=c}^{y=d} Q(x,y)\Big]_{x=a}^{x=b} dy - \int_{x=a}^{x=b} P(x,y)\Big]_{y=c}^{y=d} dx$$

$$= \int_{y=c}^{y=d} \big(Q(b,y) - Q(a,y)\big)\,dy - \int_{x=a}^{x=b} \big(P(x,d) - P(x,c)\big)\,dx$$

$$= \int_{y=c}^{y=d} Q(b,y)\,dy - \int_{y=c}^{y=d} Q(a,y)\,dy - \int_{x=a}^{x=b} P(x,d)\,dx + \int_{x=a}^{x=b} P(x,c)\,dx$$

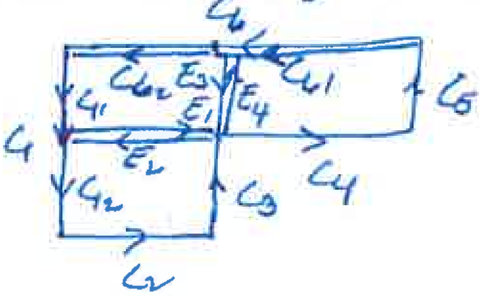
$$= \int_{c_2} Q\,dy + \int_{c_4} Q\,dy + \int_{c_3} P\,dx + \int_{c_1} P\,dx$$

$$= \int_{c_2} P\,dx + Q\,dy + \int_{c_4} P\,dx + Q\,dy + \int_{c_3} P\,dx + Q\,dy + \int_{c_1} P\,dx + Q\,dy$$

$$= \int_{\partial D} P\,dx + Q\,dy.$$

§5.1 Green's Theorem part 2

Let $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ and verify Green's theorem for a region



Solution $C_6 = C_{61} \cup C_{62}$, $C_1 = C_{11} \cup C_{12}$

$$\int_{\partial D} P\,dx + Q\,dy = \int_{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6} P\,dx + Q\,dy$$

$$= \int_{C_{11} \cup C_{12} \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_{61} \cup C_{62}} P\,dx + Q\,dy$$

$$= \int_{\substack{C_{11} \cup E_1 \cup E_4 \cup C_{62} \\ \cup C_{12} \cup C_2 \cup C_3 \cup E_2 \\ \cup C_4 \cup C_5 \cup C_{61} \cup E_3}} P\,dx + Q\,dy = \iint_{D_1 \cup D_2 \cup D_3} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

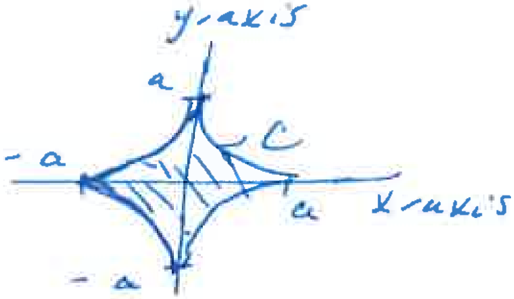
$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

In a similar manner one extends to regions of more general shapes.

<u>§5.1 Example 5</u> Find the area of the region enclosed by

$$x^{2/3}+y^{2/3}=a^{2/3} \quad (\text{where } a\in\mathbb{R}_{>0} \text{ is fixed})$$

<u>Solution:</u>



Parametrise C with $c(t)=(a\cos^3 t, a\sin^3 t)$ with $0\le t\le 2\pi$.

Then

$$(\text{Area of } D)=\frac{1}{2}\int_{\partial D} x\,dy-y\,dx=\frac{1}{2}\int_C\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

$$=\frac{1}{2}\int_{t=0}^{t=2\pi}\left(a\cos^3 t\; 3a\sin^2 t\cos t-a\sin^3 t(-3a\cos^2 t\sin t)\right)dt$$

$$=\frac{1}{2}\int_{t=0}^{t=2\pi}(3a^2\sin^2 t\cos^4 t+3a^2\sin^4 t\cos^2 t)\,dt$$

$$=\frac{1}{2}\int_{t=0}^{t=2\pi}3a^2\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)\,dt$$

$$=\frac{1}{2}\int_{t=0}^{t=2\pi}3a^2\left(\frac{1}{2}\sin 2t\right)^2 dt$$

$$=\frac{3}{8}a^2\int_{t=0}^{t=2\pi}\sin^2 2t\,dt=\frac{3}{8}a^2\int_{t=0}^{t=2\pi}\left(-\frac{1}{2}(1-2\sin^2 2t)+\frac{1}{2}\right)dt$$

$$= \frac{3}{8}a^2 \int_{t=0}^{2\pi} \left(-\tfrac{1}{2}\cos 4t + \tfrac{1}{2}\right) dt$$

$$= \frac{3}{8}a^2 \left(-\frac{1}{2}\frac{\sin 4t}{4} + \frac{1}{2}t\right)\Big]_{t=0}^{t=2\pi}$$

$$= \frac{3}{8}a^2 \left(\frac{-1}{8}\cdot 0 + \frac{1}{2}\cdot 2\pi - (0+0)\right)$$

$$= \frac{3}{8}a^2\pi.$$

<u>§5.1 Statement</u> Show that

$$\frac{1}{2}\int_{\partial D} x\,dy - y\,dx = (\text{Area of } D)$$

<u>Solution</u>

$$\frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \int_{\partial D} \frac{-1}{2}y\,dx + \frac{1}{2}x\,dy$$

$$= \iint_D \left(\frac{\partial(\frac{1}{2}x)}{\partial x} - \frac{\partial(\frac{-1}{2}y)}{\partial y}\right) dx\,dy$$

$$= \iint_D \left(\frac{1}{2} - \left(\frac{-1}{2}\right)\right) dx\,dy = \iint_D dx\,dy$$

$$= (\text{Area of } D).$$