Vector Calculus Lecture 28

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Uni Melb
A. Ram

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§5.2 Divergence theorem in the plane

Example 0: Show that

$$\iint_{\partial D} \vec{F}\cdot\hat{n}\, ds = \iint_{D} \vec{\nabla}\cdot\vec{F}\, dx\, dy$$

Solution If $\partial D = \vec{c}(t) = (x(t), y(t))$ then

$\frac{d\vec{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ is tangent to $\vec{c}(t)$ and

$\vec{n} = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$ is normal to $\vec{c}(t)$

So $$\hat{n} = \frac{1}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}\left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$$

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$ then

$$\int_{\partial D}\vec{F}\cdot\hat{n}\, ds = \int_{\partial D}(F_1\hat{\imath}+F_2\hat{\jmath})\cdot\frac{\left(\frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath}\right)}{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}\frac{ds}{dt}\, dt$$

$$= \int_{\partial D}\left(F_1\frac{dy}{dt} - F_2\frac{dx}{dt}\right)\frac{1}{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{\partial D}\left(F_1\frac{dy}{dt} - F_2\frac{dx}{dt}\right)dt = \int_{\partial D} -F_2\, dx + F_1\, dy$$

$$= \iint_D \left(\frac{\partial F_1}{\partial x} - \left(-\frac{\partial F_2}{\partial y} \right) \right) dx\,dy \qquad \left(\begin{array}{c} \text{by Green's} \\ \text{theorem} \end{array} \right)$$

$$= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx\,dy$$

$$= \iint_D \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \right) \cdot \left(F_1 \hat{i} + F_2 \hat{j} \right) dx\,dy$$

$$= \iint_D \vec{\nabla} \cdot \vec{F} \, dx\,dy$$

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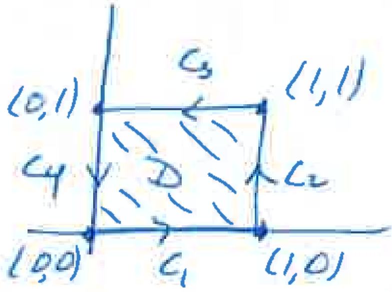
§5.2 Example 1 Let $\vec{F} = y^3\hat{\imath} + x^5\hat{\jmath}$.

Verify the divergence theorem in the plane for the region bounded by the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Solution



The divergence theorem in the plane says

$$\iint_{\partial D} \vec{F}\cdot\hat{n}\,ds = \iint_D \vec{\nabla}\cdot\vec{F}\,dx\,dy$$

For the right hand side:

$$\iint_D \vec{\nabla}\cdot\vec{F}\,dx\,dy = \iint_D \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\right)\cdot\left(y^3\hat{\imath} + x^5\hat{\jmath}\right)dx\,dy$$

$$= \iint_D (0+0)\,dx\,dy = 0.$$

For the left hand side: $\partial D = C_1 \cup C_2 \cup C_3 \cup C_4$ with

$c_1(t) = (t,0)$ with $0\le t\le 1$ and $\hat{n} = -\hat{\jmath}$, $\frac{d\vec{c}_1}{dt} = (1,0)$

$c_2(t) = (1,t)$ with $0\le t\le 1$ and $\hat{n} = \hat{\imath}$, $\frac{d\vec{c}_2}{dt} = (0,1)$

$c_3(t) = (1-t,1)$ with $0\le t\le 1$ and $\hat{n} = \hat{\jmath}$, $\frac{d\vec{c}_3}{dt} = (-1,0)$

$c_4(t) = (0,1-t)$ with $0\le t\le 1$ and $\hat{n} = -\hat{\imath}$, $\frac{d\vec{c}_4}{dt} = (0,-1)$

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So

$$\int_{\partial D} \vec{F}\cdot\hat{n}\,ds = \int_{c_1} \vec{F}\cdot\hat{n}\left|\frac{d\vec{c}_1}{dt}\right|dt + \int_{c_2} \vec{F}\cdot\hat{n}\left|\frac{d\vec{c}_2}{dt}\right|dt$$

$$+\int_{c_3} \vec{F}\cdot\hat{n}\left|\frac{d\vec{c}_3}{dt}\right|dt + \int_{c_4} \vec{F}\cdot\hat{n}\left|\frac{dc_4}{dt}\right|dt$$

$$= \int_{c_1} \vec{F}\cdot(-\hat{j})\cdot 1\,dt + \int_{c_2} \vec{F}\cdot\hat{i}\cdot 1\,dt + \int_{c_3} \vec{F}\cdot\hat{j}\cdot 1\,dt + \int_{c_4} \vec{F}\cdot(-\hat{i})\cdot 1\,dt$$

$$= \int_{c_1} -x^5\,dt + \int_{c_2} y^3\,dt + \int_{c_3} x^5\,dt + \int_{c_4} -y^3\,dt$$

$$= \int_{t=0}^{t=1} -t^5\,dt + \int_{t=0}^{t=1} t^3\,dt + \int_{t=0}^{t=1} (1-t)^5\,dt + \int_{t=0}^{t=1} -(1-t)^3\,dt$$

$$= \left.\frac{-t^6}{6}\right]_{t=0}^{t=1} + \left.\frac{t^4}{4}\right]_{t=0}^{t=1} + \left.\frac{-(1-t)^6}{6}\right]_{t=0}^{t=1} + \left.\frac{(1-t)^4}{4}\right]_{t=0}^{t=1}$$

$$= \left(-\frac{1}{6}-0\right) + \left(\frac{1}{4}-0\right) + \left(-0-\frac{-1}{6}\right) + \left(0-\frac{1}{4}\right)$$

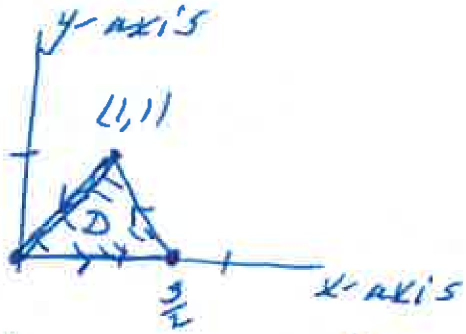
$$= 0.$$

§5.2 Example 2 let C be the triangle with vertices $(0,0)$, $(1,1)$, $(\frac{3}{2},0)$ traversed anticlockwise

let
$$\vec{F} = (2x^2y - 3x + \sin 5y,\ 5y - 2xy^2 - \cos^3 4x)$$

Evaluate $\int_C \vec{F}\cdot\hat{n}\,ds$.

Solution



Using the divergence theorem in the plane

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \vec{\nabla}\cdot\vec{F}\,dx\,dy$$

$$= \iint_D \left(\frac{\partial(2x^2y-3x+\sin 5y)}{\partial x} + \frac{\partial(5y-2xy^2-\cos^3 4x)}{\partial y}\right) dx\,dy$$

$$= \iint_D (4xy-3+0)+(5-4xy-0)\,dxdy$$

$$= \iint_D 2\,dxdy = 2\begin{pmatrix}\text{area of}\\ \text{triangle}\end{pmatrix} = 2\tfrac{1}{2}(\text{base})\cdot(\text{height})$$

$$= \frac{3}{2}\cdot 1 = \frac{3}{2}$$