Vector Calculus lecture 29

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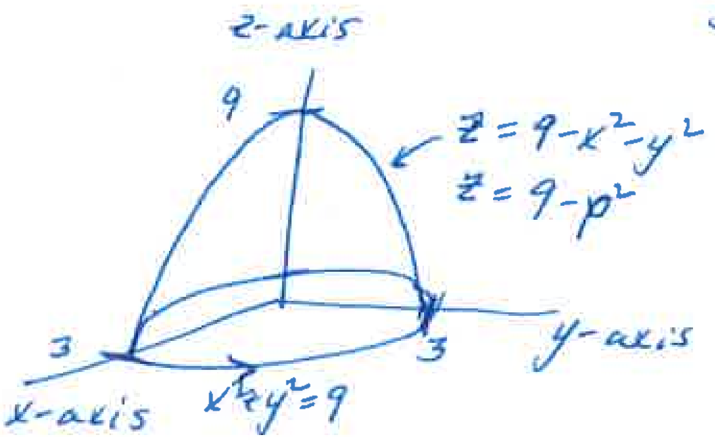
§5.3 Example 1 Verify Stokes theorem for the paraboloid

$$z = 9 - x^2 - y^2 \quad \text{with } z \geq 0$$

and $$\vec{F} = (2z - y)\hat{\imath} + (x+z)\hat{\jmath} + (3x - 2y)\hat{k}.$$

Solution: Stoke's theorem: $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$



Parametrize S using cylindrical coordinates:

$$\Phi(\rho, \varphi) = (\rho\cos\varphi, \rho\sin\varphi, 9 - \rho^2)$$

with $0 \leq \rho \leq 3$ and $0 \leq \varphi \leq 2\pi$

Parametrize ∂S by the curve

$$\vec{c}(t) = (3\cos t, 3\sin t, 0) \text{ for } 0 \leq t \leq 2\pi.$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z - y & x + z & 3x - 2y \end{vmatrix} = \begin{matrix} \hat{\imath}(-2-1) \\ -\hat{\jmath}(3-2) \\ +\hat{k}(1-(-1)) \end{matrix} = -3\hat{\imath} - \hat{\jmath} + 2\hat{k}.$$

$$\vec{T}_\rho = \left(\frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}, \frac{\partial z}{\partial \rho}\right) = (\cos\varphi, \sin\varphi, -2\rho)$$

$$\vec{T}_\varphi = \left(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\right) = (-\rho\sin\varphi, \rho\cos\varphi, 0)$$

$$\vec{T}_\rho \times \vec{T}_\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\varphi & \sin\varphi & -2\rho \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \end{vmatrix} = \begin{array}{l} \hat{\imath}(0+2\rho^2\cos\varphi) \\ -\hat{\jmath}(0-2\rho^2\sin\varphi) \\ +\hat{k}(\rho\cos^2\varphi+\rho\sin^2\varphi) \end{array}$$

$$= (2\rho^2\cos\varphi, 2\rho^2\sin\varphi, \rho)$$

(when $\varphi=0$ and $\rho=3$ this is $(2\cdot 9, 2\cdot 9\cdot 0, 3) = (18, 0, 3)$ which is pointing outwards).

Then the left hand side of Stokes' Theorem gives

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_S (-3\hat{\imath}-\hat{\jmath}+2\hat{k})\cdot(\vec{T}_\rho\times\vec{T}_\varphi)\,d\rho\,d\varphi$$

$$= \iint_S (-3\hat{\imath}-\hat{\jmath}+2\hat{k})\cdot(2\rho^2\cos\varphi\,\hat{\imath}+2\rho^2\sin\varphi\,\hat{\jmath}+\rho\,\hat{k})\,d\rho\,d\varphi$$

$$= \iint_S (-6\rho^2\cos\varphi - 2\rho^2\sin\varphi + 2\rho)\,d\varphi\,d\rho$$

$$= \int_{\rho=0}^{\rho=3}\int_{\varphi=0}^{\varphi=2\pi} (-6\rho^2\cos\varphi - 2\rho^2\sin\varphi + 2\rho)\,d\varphi\,d\rho$$

$$= \int_{\rho=0}^{\rho=3} -6\rho^2\sin\varphi + 2\rho^2\cos\varphi + 2\rho\varphi\Big]_{\varphi=0}^{\varphi=2\pi}\,d\rho$$

$$= \int_{\rho=0}^{\rho=3} (0+2\rho^2+4\rho\pi-(0+2\rho^2+0))\,d\rho$$

$$= 4\pi\frac{\rho^2}{2}\Big]_{\rho=0}^{\rho=3} = 2\pi\cdot 3^2 = 18\pi$$

The right hand side of Stokes' Theorem is

$$\int_{\partial S} \vec{F}\cdot d\vec{s} = \int_{\partial S} \left((2z-y)\hat{i} + (x+z)\hat{j} + (3x-2y)\hat{k}\right)\cdot \frac{d\vec{c}}{dt}\, dt$$

$$= \int_{\partial S} (2z-y,\ x+z,\ 3x-2y)\cdot(-3\sin t,\ 3\cos t,\ 0)\, dt$$

$$= \int_{\partial S} -3(2z-y)\sin t + 3(x+z)\cos t\ dt$$

$$= \int_{t=0}^{t=2\pi} \left(-3(2\cdot 0 - 3\sin t)\sin t + 3(3\cos t + 0)\cos t\right) dt$$

$$= \int_{t=0}^{t=2\pi} (9\sin^2 t + 9\cos^2 t)\, dt = \int_{t=0}^{t=2\pi} 9\, dt$$

$$= 9t\Big]_{t=0}^{t=2\pi} = 18\pi - 0 = 18\pi.$$

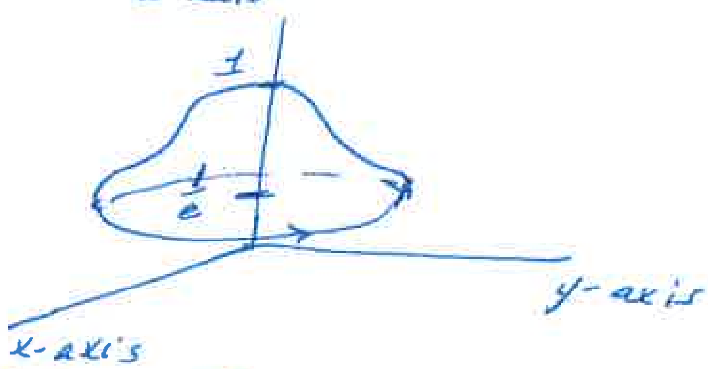
§5.3 Example 2 Evaluate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$

where $\vec{F} = (e^{y+z} - 2y, xe^{y+z}, e^{x+y})$ and

S is the surface of the bell

$$z = e^{-(x^2+y^2)} \quad \text{for } z \geq \frac{1}{e}.$$

Solution



Using Stokes' Theorem,

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}.$$

The curve ∂S is the circle $x^2+y^2=1$ with $z=\frac{1}{e}$,

$$\vec{s}(t) = (\cos t, \sin t, \tfrac{1}{e}) \quad \text{for } 0 \leq t \leq 2\pi$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} (e^{y+z} - 2y, xe^{y+z}, e^{x+y}) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_{\partial S} \left(-(e^{\sin t + \frac{1}{e}} - 2\sin t)\sin t + \cos t\, e^{\sin t + \frac{1}{e}} \cos t + 0\right) dt$$

$$= \int_{t=0}^{t=2\pi} \left(-e^{\sin t + \frac{1}{e}} \sin t + 2\sin^2 t + e^{\sin t + \frac{1}{e}} \cos^2 t\right) dt$$

$= \dots$

$$\iint_S (\vec\nabla\times\vec F)\cdot d\vec S = \int_{\partial S} \vec F\cdot d\vec s = \iint_{base} (\vec\nabla\times\vec F)\cdot d\vec S,$$ A. Rasm

where

$$\vec\nabla\times\vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y+z}-2y & xe^{y+z} & e^{x+y} \end{vmatrix} = \begin{matrix} \hat\imath\,(e^{x+y} - xe^{y+z}) \\ -\hat\jmath\,(e^{x+y} - e^{y+z}) \\ +\hat k\,(e^{y+z} - (e^{y+z} - 2)) \end{matrix}$$

$$= (e^{x+y} - xe^{y+z},\ e^{y+z} - e^{x+y},\ 2)$$

and the base is parametrized by

$$\Phi(x,y) = (x, y, \tfrac{1}{2}) \quad \text{for } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$
$$-1 \le x \le 1$$

so

$$\vec T_x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) = (1,0,0)$$
$$\vec T_y = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) = (0,1,0)$$
and $\vec T_x \times \vec T_y = \hat k$

So

$$\iint_{base} (\vec\nabla\times\vec F)\cdot d\vec S = \iint_{base} (e^{x+y} - xe^{y+z}, e^{y+z} - e^{x+y}, 2)\cdot(0,0,1)\,dx\,dy$$

$$= \iint_{base} 2\,dx\,dy = 2\left(\begin{matrix}\text{area of}\\ \text{circle}\end{matrix}\right) = 2\pi\cdot 1^2 = 2\pi$$

So

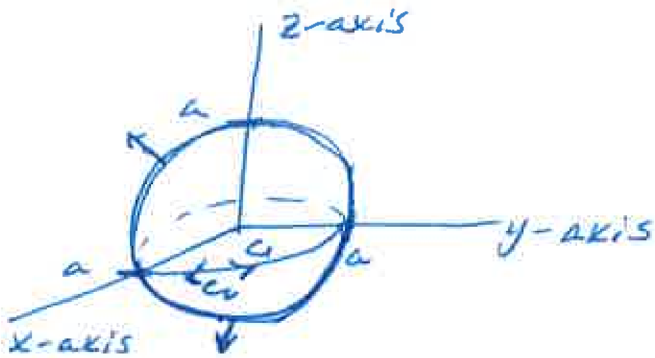
$$\iint_S (\vec\nabla\times\vec F)\cdot d\vec S = \iint_{base} (\vec\nabla\times\vec F)\cdot d\vec S = 2\pi$$

Vect. Calc. Lect. 29 04.10.2018 UniMelb A. Ram (6)

<u>§5.3 Example 3</u> Show that $\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = 0$

if S is a sphere of radius a centred at $(0,0,0)$.

<u>Solution</u>



$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_{\text{top half}} (\vec{\nabla}\times\vec{F})\cdot d\vec{S} + \iint_{\text{bottom half}} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$$= \int_{C_1} \vec{F}\cdot d\vec{s} + \int_{C_2} \vec{F}\cdot d\vec{s} = \int_{C_1} \vec{F}\cdot d\vec{s} - \int_{C_1} \vec{F}\cdot d\vec{s} = 0.$$