Vector calculus Lecture 30

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Conservative fields

$\vec{F}$ is <u>conservative</u>, or <u>irrotational</u>, or a <u>gradient field</u> if $\vec{\nabla}\times\vec{F}=0$ $(\mathrm{curl}(\vec{F})=0)$.

<u>Poincaré's theorem</u> If $\vec{F}$ is conservative then there exists φ such that $\vec{F}=\vec{\nabla}\varphi$.

<u>§5.4 Example D</u> Suppose $\vec{F}=\vec{\nabla}\varphi$. Then show that

$$\int_C \vec{F}\cdot d\vec{s} = \varphi\begin{pmatrix}\text{end point}\\ \text{of } C\end{pmatrix} - \varphi\begin{pmatrix}\text{initial point}\\ \text{of } C\end{pmatrix}$$

<u>Solution</u> If $\vec{c}(t)=(x(t),y(t),z(t))$ with $a\le t\le b$ then

$$\int_C \vec{F}\cdot d\vec{s} = \int_C \left(\vec{\nabla}\varphi\cdot\frac{d\vec{c}}{dt}\right)dt$$

$$=\int_C \left(\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j}+\frac{\partial\varphi}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}\right)dt$$

$$=\int_C\left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt}+\frac{\partial\varphi}{\partial y}\frac{dy}{dt}+\frac{\partial\varphi}{\partial z}\frac{dz}{dt}\right)dt$$

$$=\int_C \frac{d\varphi}{dt}\,dt = \int_{t=a}^{t=b}\frac{d\varphi}{dt}\,dt = \varphi\Big]_{t=a}^{t=b}$$

$$=\varphi\begin{pmatrix}\text{end point}\\ \text{of } C\end{pmatrix} - \varphi\begin{pmatrix}\text{initial point}\\ \text{of } C\end{pmatrix}.$$

§5.4 Example 1 Let $\vec{F} = x\hat{i} + y\hat{j}$.

(a) Find φ so that $\vec{\nabla}\varphi = \vec{F}$.

(b) Calculate $\int_C \vec{F}\cdot d\vec{s}$ for C on $y = x^2$ from $(0,0)$ to $(1,1)$.

(c) Calculate $\int_C \vec{F}\cdot d\vec{s}$ for C on line segments joining $(0,0)$ to $(0,1)$ to $(1,1)$.

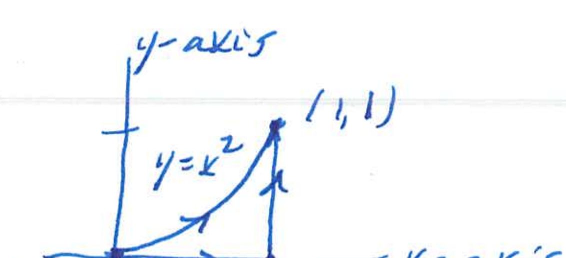
(d) Calculate $\int_C \vec{F}\cdot d\vec{s}$ for C the unit circle centred at $(0,0)$.

Solution (a) Guess: $\varphi = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

Check! $\frac{\partial\varphi}{\partial x} = \frac{1}{2}\cdot 2x = x$ and $\frac{\partial\varphi}{\partial y} = \frac{1}{2}\cdot 2y = y$

So $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} = x\hat{i} + y\hat{j} = \vec{F}$.

(b) and (c)



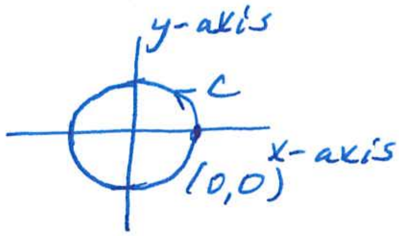
For both curves C,

$$\int_C \vec{F}\cdot d\vec{s} = \int_C \vec{\nabla}\varphi\cdot ds = \varphi\begin{pmatrix}\text{end point}\\ \text{of } C\end{pmatrix} - \varphi\begin{pmatrix}\text{initial}\\ \text{point}\\ \text{of } C\end{pmatrix}$$

$$= \varphi(1,1) - \varphi(0,0) = \left(\tfrac{1}{2}1^2 + \tfrac{1}{2}1^2\right) - \left(\tfrac{1}{2}0^2 + \tfrac{1}{2}0^2\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2} - 0 - 0 = 1.$$

(d)



$$\int_C \vec{F}\cdot d\vec{s} = \int_C \vec{\nabla}\varphi \cdot d\vec{s} = \varphi\begin{pmatrix}\text{end point}\\ \text{of } C\end{pmatrix} - \varphi\begin{pmatrix}\text{initial point}\\ \text{of } C\end{pmatrix}$$

$$= \varphi(0,0) - \varphi(0,0) = 0.$$

<u>§5.4 Example 2</u> Evaluate $\int_C \vec{F}\cdot d\vec{s}$ where

$\vec{F} = (x^2, \cos y \sin z, \sin y \cos z)$ and

$\vec{c}(t) = (t^2+1, e^t, e^{2t})$ for $0 \le t \le 1$.

<u>Solution</u>

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & \cos y \sin z & \sin y \cos z \end{vmatrix}$$

$$= \hat{i}(\cos y \cos z - \cos y \cos z) - \hat{j}(0-0) + \hat{k}(0-0) = 0.$$

So there exists φ such that $\vec{F} = \vec{\nabla}\varphi$.

<u>Guess:</u> $\varphi = \frac{1}{3}x^3 + \sin y \sin z.$

<u>Check:</u> $\frac{\partial \varphi}{\partial x} = \frac{1}{3}3x^2 + 0 = x^2,$

$\frac{\partial \varphi}{\partial y} = 0 + \cos y \sin z = \cos y \sin z$

$\frac{\partial \varphi}{\partial z} = 0 + \sin y \cos z = \sin y \cos z$

So $\vec{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k}$

$= x^2\hat{i} + \cos y \sin z\,\hat{j} + \sin y \cos z\,\hat{k} = \vec{F}.$

So

$$\int_C \vec{F}\cdot d\vec{s} = \int_C \vec{\nabla}\varphi \cdot d\vec{s} = \varphi\begin{pmatrix}\text{end point}\\ \text{of } C\end{pmatrix} - \varphi\begin{pmatrix}\text{initial point}\\ \text{of } C\end{pmatrix}$$

$= \varphi(1^2+1, e^1, e^{2\cdot 1}) - \varphi(0^2+1, e^0, e^0)$

$= \varphi(2, e, e^2) - \varphi(1,1,1)$

$= \frac{1}{3}\cdot 2^3 + \sin e \sin(e^2) - \left(\frac{1}{3}\cdot 1^3 + \sin 1 \sin 1\right)$

$= \frac{8}{3} + \sin e \sin(e^2) - \frac{1}{3} - \sin 1 \sin 1$

$= \frac{7}{3} \sin e \sin(e^2) - \sin^2 1.$