Vector Calculus Lecture 31 — 09.10.2018 — UniMelb — A. Ram — ①

§5.5 Gauss' divergence theorem

$$\iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\,dV = \iint_{\partial\Omega} \vec{F}\cdot d\vec{S}$$

Example 1 Let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and S the unit sphere centred at the origin. Evaluate

$$\iint_S \vec{F}\cdot d\vec{S}.$$

Solution

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_{\Omega} (\vec{\nabla}\cdot\vec{F})\,dV = \iiint_{\Omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)dV$$

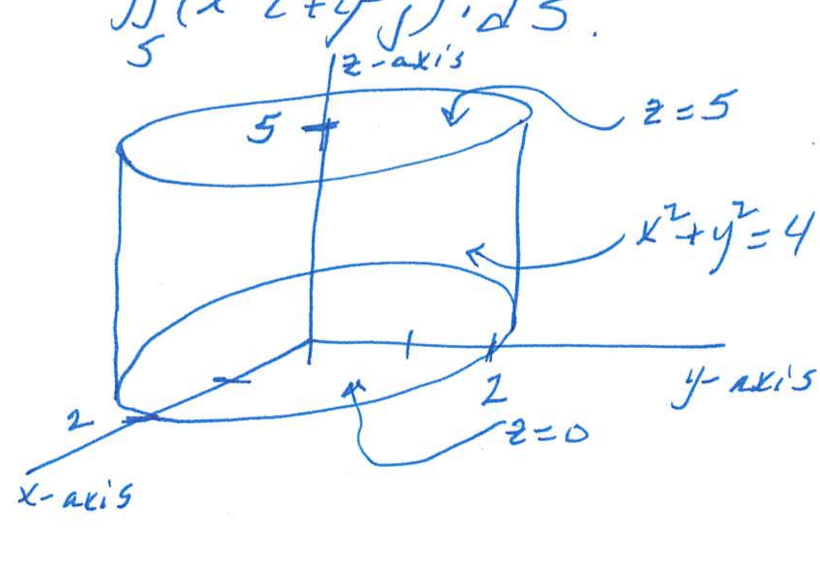
$$= \iiint_{\Omega} 3\,dV = 3\cdot\begin{pmatrix}\text{Volume of}\\ \text{sphere}\end{pmatrix} = 3\cdot\frac{4}{3}\pi\cdot 1^3 = 4\pi.$$

Example 2 Let S be the closed cylinder

$x^2+y^2=4$, $z=0$ and $z=5$.

Evaluate

$$\iint_S (x^3\hat{i} + y^3\hat{j})\cdot d\vec{S}.$$



Solution Use Gauss' divergence theorem,

$$\iint_S (x^3\hat{\imath}+y^3\hat{\jmath})\cdot d\vec{S} = \iiint_\Omega \vec{\nabla}\cdot(x^3\hat{\imath}+y^3\hat{\jmath})\,dV$$

$$= \iiint_\Omega \left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)\cdot(x^3\hat{\imath}+y^3\hat{\jmath})\,dV$$

$$= \iiint_\Omega \left(\frac{\partial x^3}{\partial x}+\frac{\partial y^3}{\partial y}+0\right)dV = \iiint_\Omega (3x^2+3y^2)\,dV.$$

Using cylindrical coordinates

$$\iiint_\Omega (3x^2+3y^2)\,dV = \iiint_\Omega 3\rho^2\,dV = \iiint_\Omega 3\rho^2\,\rho\,d\rho\,d\varphi\,dz$$

$$= \int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=2} 3\rho^3\,d\rho\,d\varphi\,dz$$

$$= \int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi} \left.3\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=2} d\varphi\,dz$$

$$= \int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi} \left(\frac{3\cdot 2^4}{4}-0\right)d\varphi\,dz$$

$$= \int_{z=0}^{z=5} \left.3\cdot 4\varphi\right]_{\varphi=0}^{\varphi=2\pi} dz = \int_{z=0}^{z=5}(12\cdot 2\pi-0)\,dz$$

$$= \left.24\pi z\right]_{z=0}^{z=5} = 24\cdot 5\pi - 0 = 120\pi.$$

§5.5 Example 3 Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, where

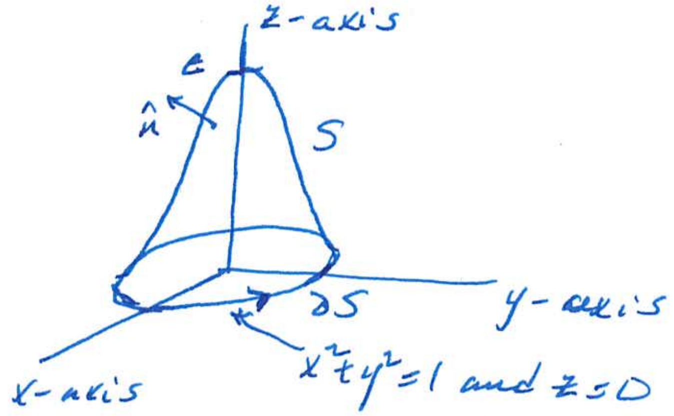
S is the surface of the bell

$$z = (1-x^2-y^2)\,e^{1-x^2-3y^2} \quad \text{for } z \geq 0$$

and

$$\vec{F} = (e^y \cos z,\ (x^3+1)^{\frac{1}{2}} \sin z,\ x^2+y^2+3)$$

Solution



Let V be the solid region bounded by S and the plane $z=0$. Then

$$\partial V = S \cup (\text{base circle}).$$

So

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial V} \vec{F}\cdot d\vec{S} - \iint_{\text{base}} \vec{F}\cdot d\vec{S}$$

$$= \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV - \iint_{\text{base}} \vec{F}\cdot d\vec{S}, \quad \text{by the divergence theorem}$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz - \iint_{\text{base}} \vec{F}\cdot\hat{n}\,dS$$

$$= \iiint_V (0+0+0)\,dx\,dy\,dz - \iint_{base} \vec{F}\cdot\hat{n}\,dS$$

$$= -\iint_{base} \vec{F}\cdot(-\hat{k})\,dS,$$ since the outward pointing normal for V is $-\hat{k}$ on the base

$$= -\iint_{base} -(x^2+y^2+3)\,dS$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (r^2+3)\,r\,dr\,d\theta,$$ since the base is a circle of radius 1 in the xy-plane

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (r^3+3r)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left.\frac{r^4}{4}+\frac{3r^2}{2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{4}+\frac{3}{2}-(0+0)\right) d\theta$$

$$= \left.\frac{7}{4}\theta\right]_{\theta=0}^{\theta=2\pi} = \frac{7}{4}(2\pi-0) = \frac{7}{2}\pi.$$

§5.5 Example Let

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$ and let Ω

be a region to which Gauss' theorem applies. Show that

$$\frac{1}{3}\iint_{\partial\Omega} \vec{F}\cdot d\vec{S} = (\text{volume of } \Omega)$$

Solution: Using the divergence theorem

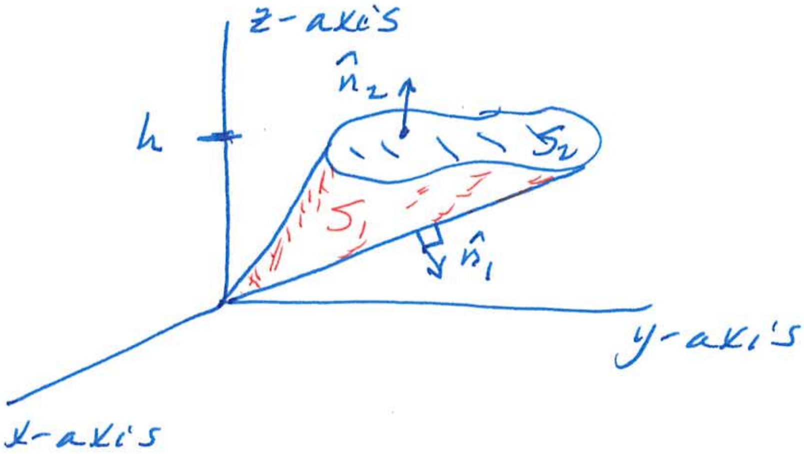
$$\frac{1}{3}\iint_{\partial\Omega} \vec{F}\cdot d\vec{S} = \frac{1}{3}\iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\, dV$$

$$= \frac{1}{3}\iiint_{\Omega}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right) dV$$

$$= \frac{1}{3}\iiint_{\Omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dV = \frac{1}{3}\iiint_{\Omega} 3\, dV$$

$$= \iiint_{\Omega} dV = (\text{Volume of } V).$$

§5.5 Example 4 Find the volume of a solid cone with base area A and height h.



Solution Using the divergence theorem

$$\text{Volume of cone} = \frac{1}{3}\iint_{\substack{\text{boundary}\\ \text{of cone}}} (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot d\vec{S}$$

$$= \frac{1}{3}\iint_{S_1} (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot d\vec{S} + \frac{1}{3}\iint_{S_2} (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot d\vec{S}$$

$$= \frac{1}{3}\iint_{S_1} (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot \hat{n}_1\, dS + \frac{1}{3}\iint_{S_2} (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot \hat{n}_2\, d\vec{S}$$

$$= \frac{1}{3}\iint_{S_1} 0\, dS + \frac{1}{3}\iint_{S_2} (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot \hat{k}\, dS$$

$$= \frac{1}{3}\iint_{S_2} z\, dS = \frac{1}{3}\iint_{S_2} h\, dS = \frac{h}{3}\iint_{S_2} dS$$

$$= \frac{h}{3}\left(\text{area of } S_2\right) = \frac{h}{3}A.$$