Vector Calculus lecture 33 — 12.10.2018, UniMelb, A. Ram — ①

§6.2 Example 1 Let

$$f(r,\theta,\varphi) = r\theta\varphi \quad\text{and}\quad \vec{F} = \sin\theta\,\hat{r} + r\hat{\varphi}.$$

Find $\vec{\nabla}f$, $\vec{\nabla}\cdot\vec{F}$, $\vec{\nabla}\times\vec{F}$, $\nabla^2 f$

in spherical coordinates.

Solution For spherical coordinates

$$h_r = 1,\quad h_\theta = r,\quad h_\varphi = r\sin\theta$$

From formula 1 on formula sheet 3,

$$\vec{\nabla}f = \frac{1}{h_r}\frac{\partial f}{\partial r}\hat{r} + \frac{1}{h_\theta}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{h_\varphi}\frac{\partial f}{\partial\varphi}\hat{\varphi}$$

$$= \frac{1}{1}\frac{\partial(r\theta\varphi)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial(r\theta\varphi)}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial(r\theta\varphi)}{\partial\varphi}\hat{\varphi}$$

$$= \theta\varphi\,\hat{r} + \frac{r\varphi}{r}\hat{\theta} + \frac{r\theta}{r\sin\theta}\hat{\varphi}$$

$$= \theta\varphi\,\hat{r} + \varphi\,\hat{\theta} + \frac{\theta}{\sin\theta}\hat{\varphi}$$

From formula 2 on formula sheet 3,

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{h_r h_\theta h_\varphi}\left(\frac{\partial(h_\theta h_\varphi F_r)}{\partial r} + \frac{\partial(h_r h_\varphi F_\theta)}{\partial\theta} + \frac{\partial(h_r h_\theta F_\varphi)}{\partial\varphi}\right)$$

$$= \frac{1}{1\cdot r\cdot r\sin\theta}\left(\frac{\partial(r\,r\sin\theta\sin\theta)}{\partial r} + \frac{\partial(1\cdot r\sin\theta\cdot 0)}{\partial\theta} + \frac{\partial(1\cdot r\cdot r)}{\partial\varphi}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(2r\sin^2\theta + 0 + 0\right) = \frac{2}{r}\sin\theta.$$

From formula 3 on Formula sheet 3,

$$\vec{\nabla}\times\vec{F} = \frac{1}{h_r h_\theta h_\varphi}\begin{vmatrix} h_r\hat{r} & h_\theta\hat{\theta} & h_\varphi\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ h_r F_r & h_\theta F_\theta & h_\varphi F_\varphi \end{vmatrix}$$

$$= \frac{1}{1\cdot r\cdot r\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ \sin\theta & 0 & r\sin\theta\, r \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{pmatrix} \hat{r}\,(r^2\cos\theta - 0) \\ -r\hat{\theta}\,(2r\sin\theta - 0) \\ + r\varphi\sin\theta\,(0-\cos\theta) \end{pmatrix}$$

$$= \cot\theta\,\hat{r} - 2\hat{\theta} - \frac{1}{r}\cos\theta\,\hat{\varphi}.$$

From formula 4 on Formula Sheet 3

$$\nabla^2 f = \frac{1}{h_r h_\theta h_\varphi}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_\varphi}{h_r}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_r h_\varphi}{h_\theta}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_r h_\theta}{h_\varphi}\frac{\partial f}{\partial\varphi}\right)\right)$$

$$= \frac{1}{1\cdot r\cdot r\sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{r\cdot r\sin\theta}{1}\cdot\theta\varphi\right) + \frac{\partial}{\partial\theta}\left(\frac{1\cdot r\sin\theta}{r}\cdot r\varphi\right) + \frac{\partial}{\partial\varphi}\left(\frac{1\cdot r}{r\sin\theta}\cdot r\theta\right)\right)$$

$$= \frac{1}{r^2 \sin\theta}\left(2r\,\theta(\sin\theta)\varphi + r\varphi\cos\theta + 0\right)$$

$$= \frac{2}{r}\theta\varphi + \frac{1}{r}\varphi\cot\theta.$$

<u>§6.2 Example 3</u> Find $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right)$, where

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2+y^2+z^2}$.

<u>Solution</u> $\hat{r} = \frac{\vec{r}}{r}$ and $\frac{\vec{r}}{r^2} = \frac{1}{r}\hat{r} = \frac{1}{r}\hat{r} + 0\hat{\theta} + 0\hat{\varphi}$

So, by formula 2 on Formula sheet 3,

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right) = \vec{\nabla}\cdot\left(\frac{1}{r}\hat{r}\right)$$

$$= \frac{1}{h_r h_\varphi h_\theta}\left(\frac{\partial}{\partial r}(h_\theta h_\varphi F_r) + \frac{\partial}{\partial\theta}(h_r h_\varphi F_\theta) + \frac{\partial}{\partial\varphi}(h_r h_\theta F_\varphi)\right)$$

$$= \frac{1}{1\cdot r\cdot r\sin\theta}\left(\frac{\partial}{\partial r}\left(r\cdot r\sin\theta\,\frac{1}{r}\right) + \frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial\varphi}(0)\right)$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r\sin\theta) = \frac{1}{r^2\sin\theta}\cdot\sin\theta = \frac{1}{r^2}$$

<u>§6.2 Example 2</u> Express

$\frac{\partial u}{\partial t} = k\,\nabla^2 u$, in spherical coordinates

if k is a constant and u depends only on r and t.

<u>Solution</u> From formula 4 on Formula sheet 3,

$$\frac{\partial u}{\partial t} = k\,\vec{\nabla}^2 u$$

$$= k\,\frac{1}{h_r h_\theta h_\varphi}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_\varphi}{h_r}\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{h_r h_\varphi}{h_\theta}\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{h_r h_\theta}{h_\varphi}\frac{\partial u}{\partial \varphi}\right)\right)$$

$$= k\,\frac{1}{1\cdot r\cdot r\sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{r\cdot r\sin\theta}{1}\frac{\partial u}{\partial r}\right) + 0 + 0\right)$$

$$= \frac{k}{r^2\sin\theta}\,\frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial u}{\partial r}\right)$$

$$= \frac{k}{r^2\sin\theta}\left(2r\sin\theta\,\frac{\partial u}{\partial r} + r^2\sin\theta\,\frac{\partial^2 u}{\partial r^2}\right)$$

$$= \frac{2k}{r}\,\frac{\partial u}{\partial r} + k\,\frac{\partial^2 u}{\partial r^2}.$$

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§6.2 Example 4 Find $\nabla^2(r^2\log r)$,

where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = \sqrt{x^2+y^2+z^2}$

Solution Using spherical coordinates and formula 4 from Formula sheet 3

$$\vec{\nabla}^2(r^2\log r) = \frac{1}{h_r h_\theta h_\varphi}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_\varphi}{h_r}\frac{\partial(r^2\log r)}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_r h_\varphi}{h_\theta}\frac{\partial(r^2\log r)}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_r h_\theta}{h_\varphi}\frac{\partial(r^2\log r)}{\partial\varphi}\right)\right)$$

$$= \frac{1}{1\cdot r\cdot r\sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{r\cdot r\sin\theta}{1}\left(2r\log r + r^2\frac{1}{r}\right)\right)+0+0\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(2r^3\log r\sin\theta + r^3\sin\theta\right)\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\sin\theta\left(6r^2\log r + 2r^3\frac{1}{r}\right) + 3r^2\sin\theta\right)$$

$$= 6\log r + 2 + 3 = 6\log r + 5.$$