Vect. Calc. Lecture 34 — 16.10.2018, UniMelb, A. Ram — ①

Length, area and volume differentials

(1) If

$\vec{c}(t) = (u(t), v(t), w(t))$ in u, v, w coordinates

$$d\vec{s} = \frac{d\vec{c}}{dt}\cancel{dt}\left(h_u \frac{du}{dt}\hat{u} + h_v \frac{dv}{dt}\hat{v} + h_w \frac{dw}{dt}\hat{w}\right) dt$$

$$ds = \left|\frac{d\vec{c}}{dt}\right| dt = \sqrt{h_u^2\left(\frac{du}{dt}\right)^2 + h_v^2\left(\frac{dv}{dt}\right)^2 + h_w\left(\frac{dw}{dt}\right)^2}\; dt.$$

(2) $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ in x,y,z coordinates

$$d\vec{S} = h_u h_v (\hat{u} \times \hat{v})\, du\, dv$$

$$dS = h_u h_v |\hat{u} \times \hat{v}|\, du\, dv$$

and $dS = h_u h_v\, du\, dv$ if $\hat{u}, \hat{v}$ are orthogonal.

(3) $dV = |\text{Jacobian}|\, du\, dv\, dw$

So $$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$

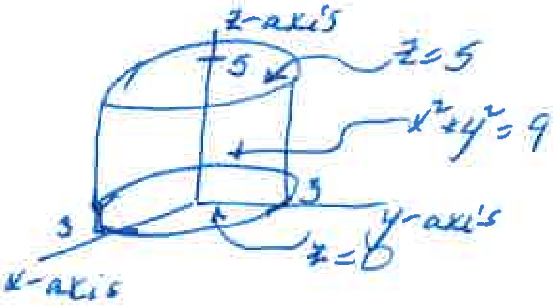
So $$dV = h_u h_v h_w\, |\hat{u} \cdot (\hat{v} \times \hat{w})|\, du\, dv\, dw$$

and $dV = h_u h_v h_w\, du\, dv\, dw$ if $\hat{u}, \hat{v}, \hat{w}$ are orthogonal

<u>§6.3 Example 1</u> Evaluate $\iint_S z\,dS$ for the cylinder

$$x^2+y^2=9 \text{ and } 0 \le z \le 5$$

<u>Solution:</u>



Cylindrical coordinates with $\rho = 3$.

$$\iint_S z\,dS = \iint_S z\,d\varphi\,dz\,h_\varphi h_z = \iint_S z\,d\varphi\,dz \cdot \rho \cdot 1$$

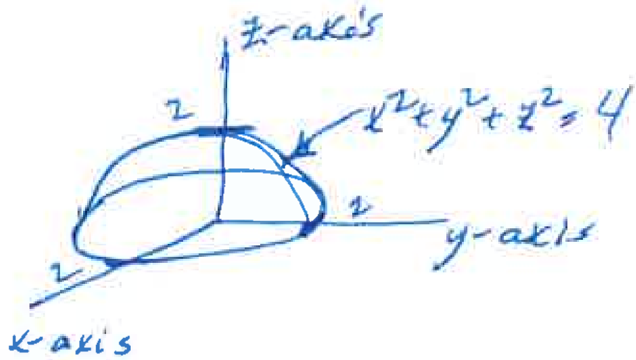
$$= \int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi} z\cdot 3\,d\varphi\,dz = \int_{z=0}^{z=5} 3z\varphi\Big]_{\varphi=0}^{\varphi=2\pi} dz = \int_{z=0}^{z=5} 6\pi z\,dz$$

$$= 3\pi z^2\Big]_{z=0}^{z=5} = 3\pi(25-0) = 75\pi$$

<u>§6.3 Example 2</u> Evaluate $\iint_S z\,dS$ for the hemisphere

$$x^2+y^2+z^2=4 \text{ with } z \ge 0.$$

<u>Solution</u>



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Spherical coordinates with $r=2$. UniMelb
A. Ram

③

$$\iint_S z\,dS = \iint_S z h_\varphi h_\theta \, d\varphi\, d\theta = \iint_S z\, r\sin\theta\, r\, d\varphi\, d\theta$$

$$= \iint_S 2^2\, r\cos\theta \sin\theta\, d\varphi\, d\theta = \iint_S 8 \sin\theta\cos\theta\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{\varphi=0}^{\varphi=2\pi} 4\sin 2\theta\, d\varphi\, d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} 4\sin 2\theta\ \varphi\Big]_{\varphi=0}^{\varphi=2\pi} d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} 4\sin 2\theta \cdot 2\pi\, d\theta = -4\pi\cos 2\theta\Big]_{\theta=0}^{\theta=\pi/2}$$

$$= -4\pi(\cos\pi - \cos 0) = -4\pi(-1-1) = 8\pi.$$

<u>Orange problem book 62(a)</u>

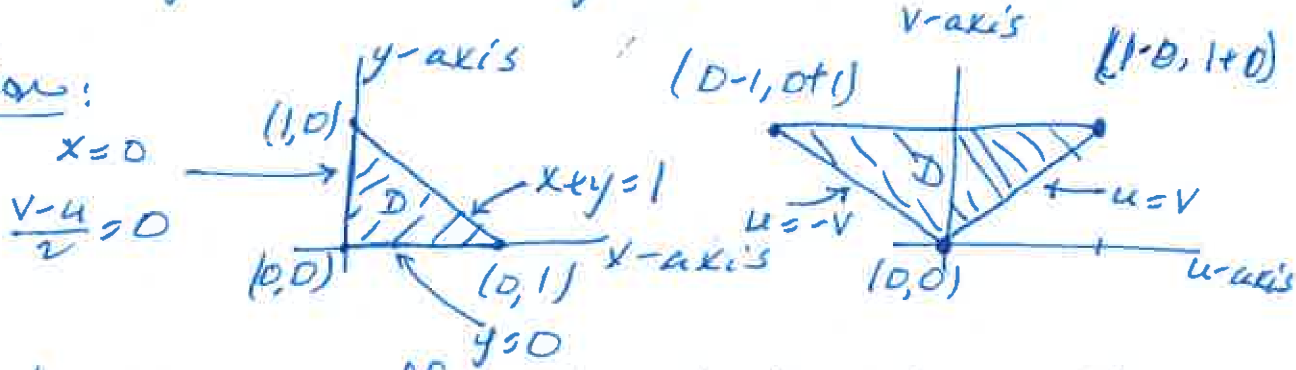
Let D be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

Evaluate $\displaystyle\iint_D \exp\left(\frac{y-x}{y+x}\right) dx\,dy$

by making the substitution

$u = y - x$ and $v = y + x$.

<u>Solution:</u>



$$\iint_D \exp\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_D \exp\left(\frac{u}{v}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_D e^{u/v} \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} du\,dv = \iint_D e^{u/v} \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} du\,dv$$

$$= \iint_D e^{u/v} \frac{1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} du\,dv = \iint_D e^{u/v} \frac{1}{|-1-1|} du\,dv$$

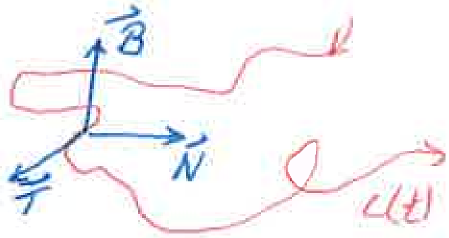
$$= \int_{v=0}^{v=1} \int_{u=-v}^{u=v} e^{u/v} \tfrac{1}{2}\, du\,dv = \int_{v=0}^{v=1} \tfrac{1}{2} v e^{u/v} \Big]_{u=-v}^{u=v} dv$$

$$= \int_{v=0}^{v=1} \tfrac{1}{2} v \left(e^{v/v} - e^{-v/v}\right) dv = \int_{v=0} \frac{(e-e^{-1})}{2} v\,dv = \frac{(e-e^{-1})}{2} \frac{v^2}{2}\Big]_{v=0}^{v=1} = \frac{(e-e^{-1})}{4}.$$

Review of $\vec{T}$, $\vec{N}$ and $\vec{B}$

$$\frac{ds}{dt} = \left|\frac{d\vec{c}}{dt}\right|$$

$$\vec{T} = \frac{\frac{d\vec{c}}{dt}}{\left|\frac{d\vec{c}}{dt}\right|} = \frac{d\vec{c}}{ds}, \qquad \vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} \qquad \vec{B} = \vec{T} \times \vec{N}$$



So $\vec{T}\times\vec{N} = \vec{B}$, $\vec{N}\times\vec{B} = \vec{T}$, $\vec{B}\times\vec{T} = \vec{N}$.

Example (a) Show that $\frac{d\vec{B}}{ds}$ is orthogonal to $\vec{B}$ and $\vec{T}$

(b) Show that

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}, \quad \frac{d\vec{B}}{ds} = -\tau\vec{N}, \quad \frac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T}.$$

Solution (a) $0 = \frac{d(\vec{B}\cdot\vec{B})}{ds} = \frac{d\vec{B}}{ds}\cdot\vec{B} + \vec{B}\cdot\frac{d\vec{B}}{ds} = 2\vec{B}\cdot\frac{d\vec{B}}{ds}$

and

$$0 = \frac{d(\vec{B}\cdot\vec{T})}{ds} = \frac{d\vec{B}}{ds}\cdot\vec{T} + \vec{B}\cdot\frac{d\vec{T}}{ds} = \frac{d\vec{B}}{ds}\cdot\vec{T} + \vec{B}\cdot\kappa\vec{N}$$

$$= \frac{d\vec{B}}{ds}\cdot\vec{T} + 0 = \frac{d\vec{B}}{ds}\cdot\vec{T}.$$

So $\frac{d\vec{B}}{ds}$ is orthogonal to $\vec{B}$ and $\vec{T}$.

So $\frac{d\vec{B}}{ds}$ is parallel to $\vec{N}$.

(b) $$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \kappa \vec{N}.$$

$\frac{d\vec{B}}{ds} = -\tau \vec{N}$, by definition of the torsion τ.

$$\frac{d\vec{N}}{ds} = \frac{d(\vec{B}\times\vec{T})}{ds} = \frac{d\vec{B}}{ds}\times\vec{T} + \vec{B}\times\frac{d\vec{T}}{ds}$$

$$= -\tau\vec{N}\times\vec{T} + \vec{B}\times\kappa\vec{N}$$

$$= (-\tau)(-\vec{B}) + \kappa(\vec{B}\times\vec{N})$$

$$= \tau\vec{B} - \kappa\vec{T}$$