Vector Calculus Lecture 36 19.10.2018 UniMelb A. Ram ①

§5.3 Example ① Let S be a surface

$$\Phi(x,y) = (x, y, f(x,y))$$

with boundary the curve ∂S given by

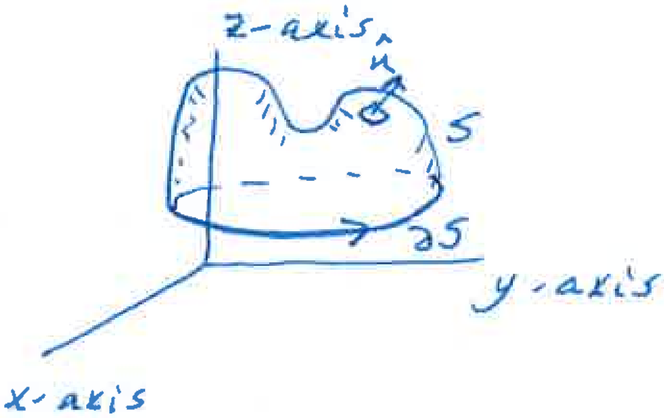
$$\vec{c}(t) = (x(t), y(t), z(t))$$

and let

$$\vec{F} = F_1 \hat{\imath} + F_2 \hat{\jmath} + F_3 \hat{k}.$$

Verify Stokes' theorem.

Solution Stokes' theorem is $\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$.



Right hand side: $d\vec{S} = (\vec{T}_x \times \vec{T}_y)\,dx\,dy$

$$\vec{T}_x = \left(1, 0, \frac{\partial f}{\partial x}\right) \quad \text{and} \quad \vec{T}_y = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

$$\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \begin{matrix} \hat{\imath}\left(0 - \frac{\partial f}{\partial x}\right) \\ -\hat{\jmath}\left(\frac{\partial f}{\partial y} - 0\right) \\ +\hat{k}(1-0) \end{matrix} = -\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k}$$

=

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{matrix} \hat{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \\ -\hat{\jmath}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) \\ +\hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \end{matrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

So

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_S \begin{pmatrix} -\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\frac{\partial f}{\partial x} \\ -\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\frac{\partial f}{\partial y} \\ +\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \end{pmatrix} dx\,dy$$

$$= \iint_S \begin{pmatrix} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} - \frac{\partial F_3}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial f}{\partial x} \\ - \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_3}{\partial x}\frac{\partial f}{\partial y} \end{pmatrix} dx\,dy$$

<u>Left Hand side</u> $d\vec{s} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$

$$\int_{\partial S} \vec{F}\cdot d\vec{s} = \int_{\partial S} (F_1, F_2, F_3)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

$$= \int_{\partial S}\left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$$

$$= \int_{\partial S} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) \right) dt$$

$$= \int_{\partial S} \left(\left(F_1 + F_3 \frac{\partial f}{\partial x} \right) \frac{dx}{dt} + \left(F_2 + F_3 \frac{\partial f}{\partial y} \right) \frac{dy}{dt} \right) dt$$

$$= \int_{\partial S} \left(F_1 + F_3 \frac{\partial f}{\partial x} \right) dx + \left(F_2 + F_3 \frac{\partial f}{\partial y} \right) dy$$

$$= \iint_S \left(\frac{\partial}{\partial x}\left(F_2 + F_3 \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y}\left(F_1 + F_3 \frac{\partial f}{\partial x} \right) \right) dx\,dy \quad \left(\text{by Greens theorem} \right)$$

$$= \iint_S \left(\begin{aligned} &\left(\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial x} + \cancelto{0}{\frac{\partial F_2}{\partial y}\frac{\partial y}{\partial x}} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} \right) \\ &+ \left(\frac{\partial F_3}{\partial x}\frac{\partial x}{\partial x} + \cancelto{0}{\frac{\partial F_3}{\partial y}\frac{\partial y}{\partial x}} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial x} \right)\frac{\partial f}{\partial y} \\ &+ F_3 \frac{\partial^2 f}{\partial x \partial y} \\ &- \left(\cancelto{0}{\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial y}} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} \right) \\ &- \left(\cancelto{0}{\frac{\partial F_3}{\partial x}\frac{\partial x}{\partial y}} + \frac{\partial F_3}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial z}{\partial y} \right)\frac{\partial f}{\partial x} \\ &- F_3 \frac{\partial^2 f}{\partial y \partial x} \end{aligned} \right) dx\,dy$$

since $z = f$

$$= \iint_S \left(\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial F_3}{\partial y}\frac{\partial f}{\partial x} \right) dx\,dy$$

$$= \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} - \frac{\partial F_3}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_3}{\partial x}\frac{\partial f}{\partial y} \right) dx\,dy$$

<u>§5.4 Example O</u> Let

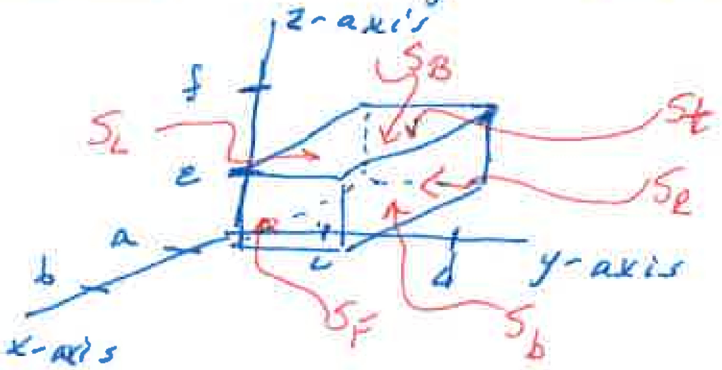
$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}.$$

Let V be the rectangular region bounded by the planes

$$x=a,\ x=b,\ y=c,\ y=d,\ z=e,\ z=f$$

where $a,b,c,d,e,f \in \mathbb{R}$ with $a<b$, $c<d$ and $e<f$. Verify the Divergence theorem.

<u>Solution</u> The Divergence theorem is $\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$.



$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_{S_F} \vec{F}\cdot\hat{n}\,dS + \iint_{S_B} \vec{F}\cdot\hat{n}\,dS + \iint_{S_L} \vec{F}\cdot\hat{n}\,dS + \iint_{S_R} \vec{F}\cdot\hat{n}\,dS + \iint_{S_b} \vec{F}\cdot\hat{n}\,dS + \iint_{S_t} \vec{F}\cdot\hat{n}\,dS$$

$$= \iint_{S_F} (\vec{F}\cdot\hat{i})\,dS + \iint_{S_B} \vec{F}\cdot(-\hat{i})\,dS$$

$$+ \iint_{S_L} \vec{F}\cdot(-\hat{j})\,dS + \iint_{S_R} \vec{F}\cdot\hat{j}\,dS$$

$$+ \iint_{S_b} \vec{F}\cdot(-\hat{k})\,dS + \iint_{S_t} \vec{F}\cdot\hat{k}\,dS$$

$$= \iint_{S_F} F_1\,dS + \iint_{S_B} -F_1\,dS + \iint_{S_L} -F_2\,dS + \iint_{S_R} F_2\,dS$$

$$+ \iint_{S_b} -F_3\,dS + \iint_{S_t} F_3\,dS$$

$$= \iint_{S_F} F_1(b,y,z)\,dy\,dz + \iint_{S_B} -F(a,y,z)\,dy\,dz$$

$$+ \iint_{S_L} -F_2(x,c,z)\,dx\,dz + \iint_{S_R} F_2(x,d,z)\,dx\,dz$$

$$+ \iint_{S_b} -F_3(x,y,e)\,dx\,dy + \iint_{S_t} F_3(x,y,f)\,dx\,dy$$

$$= \int_{z=e}^{z=f}\int_{y=c}^{y=d} \big(F_1(b,y,z) - F_1(a,y,z)\big)\,dy\,dz$$

$$+ \int_{z=e}^{z=f}\int_{x=a}^{x=b} \big(F_2(x,d,z) - F_2(x,c,z)\big)\,dx\,dz$$

$$+ \int_{y=c}^{y=d}\int_{x=a}^{x=b} \big(F_3(x,y,f) - F_3(x,y,e)\big)\,dx\,dy$$

$$= \int_{z=e}^{z=f}\int_{y=c}^{y=d}\int_{x=a}^{x=b} \frac{\partial F_1}{\partial x}(x,y,z)\,dx\,dy\,dz$$

$$+ \int_{z=e}^{z=f}\int_{x=a}^{x=b}\int_{y=c}^{y=d} \frac{\partial F_2}{\partial y}(x,y,z)\,dx\,dy\,dz + \int_{y=c}^{y=d}\int_{x=a}^{x=b}\int_{z=e}^{z=f} \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz$$

$$= \iiint_V (\vec{\nabla} \cdot \vec{F})\, dV.$$

<u>Note</u>

To prove the divergence theorem for a <u>general</u> region V, divide V into a union of rectangular boxes and apply the previous calculation.

§6.2 Example ① Assume $\hat{u}, \hat{v}, \hat{w}$ are orthogonal. Compute $\vec{\nabla} f$ in terms of $\hat{u}, \hat{v}, \hat{w}$.

Solution Since $\hat{u}, \hat{v}, \hat{w}$ are orthogonal,

$$\hat{\imath} = (\hat{\imath}\cdot\hat{u})\hat{u} + (\hat{\imath}\cdot\hat{v})\hat{v} + (\hat{\imath}\cdot\hat{w})\hat{w}$$
$$\hat{\jmath} = (\hat{\jmath}\cdot\hat{u})\hat{u} + (\hat{\jmath}\cdot\hat{v})\hat{v} + (\hat{\jmath}\cdot\hat{w})\hat{w}$$
$$\hat{k} = (\hat{k}\cdot\hat{u})\hat{u} + (\hat{k}\cdot\hat{v})\hat{v} + (\hat{k}\cdot\hat{w})\hat{w}$$

So

$$\begin{aligned}
\vec{\nabla} f &= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \\
&= \frac{\partial f}{\partial x}\left((\hat{\imath}\cdot\hat{u})\hat{u} + (\hat{\imath}\cdot\hat{v})\hat{v} + (\hat{\imath}\cdot\hat{w})\hat{w}\right) \\
&\quad + \frac{\partial f}{\partial y}\left((\hat{\jmath}\cdot\hat{u})\hat{u} + (\hat{\jmath}\cdot\hat{v})\hat{v} + (\hat{\jmath}\cdot\hat{w})\hat{w}\right) \\
&\quad + \frac{\partial f}{\partial z}\left((\hat{k}\cdot\hat{u})\hat{u} + (\hat{k}\cdot\hat{v})\hat{v} + (\hat{k}\cdot\hat{w})w\right) \\
&= \frac{\partial f}{\partial x}\left(\frac{1}{h_u}\frac{\partial x}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial x}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial x}{\partial w}\hat{w}\right) \\
&\quad + \frac{\partial f}{\partial y}\left(\frac{1}{h_u}\frac{\partial y}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial y}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial y}{\partial w}\hat{w}\right) \\
&\quad + \frac{\partial f}{\partial z}\left(\frac{1}{h_u}\frac{\partial z}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial z}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial z}{\partial w}\hat{w}\right) \\
&= \frac{1}{h_u}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\right)\hat{u} \\
&\quad + \frac{1}{h_v}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\right)\hat{v} \\
&\quad + \frac{1}{h_w}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}\right)\hat{w} =
\end{aligned}$$

$$= \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}.$$