Vector calculus: Lecture 3

The function f is continuous at a if f satisfies:

$$\lim_{x\to a} f(x) = f(a)$$

The function f is C^r at a if f satisfies

$$\left.\frac{\partial^r f}{\partial x_{i_1}\cdots\partial x_{i_r}}\right|_{x=a}$$ exist and are continuous.

The differentiable / C^r theorem

$$\cdots \Rightarrow \begin{array}{c} f \text{ is } C^2 \\ \text{at } a \end{array} \Rightarrow \begin{array}{c} f \text{ is } C^1 \\ \text{at } a \end{array} \Rightarrow \begin{array}{c} f \text{ is differentiable} \\ \text{at } a \end{array} \Rightarrow \left.\frac{\partial f}{\partial x_j}\right|_{x=a} \text{ exist}$$

Conceptually,

$f: \mathbb{R}^2 \to \mathbb{R}^1$ is differentiable at (a_1, a_2)

if the graph of f has a tangent plane at (a_1, a_2).

Example / §1.3 Let $f(x,y) = x^2+y^2$.
Where is f differentiable?

Solution:

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y.$$

Since $2x$ and $2y$ are polynomials they are continuous for $(x,y) \in \mathbb{R}^2$.

So, by the differentiability theorem
$f(x,y) = x^2+y^2$ is differentiable
for $(x,y) \in \mathbb{R}^2$.

§1.3 Example 2. Let

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Is f differentiable?

Solution: First compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

When $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2(x^2+y^2)^{-\frac{1}{2}}\right)$$

$$= x^2\left(\tfrac{-1}{2}\right)(x^2+y^2)^{-3/2}2x + 2x(x^2+y^2)^{-\frac{1}{2}}$$

$$= \frac{-x^3}{(x^2+y^2)^{3/2}} + \frac{2x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{-x^3+2x(x^2+y^2)}{(x^2+y^2)^{3/2}}$$

$$= \frac{x^3+2y^2x}{(x^2+y^2)^{3/2}} = \frac{x^3+2xy^2}{(x^2+y^2)^{3/2}} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2(x^2+y^2)^{-\frac{1}{2}}\right) = x^2\left(\tfrac{-1}{2}\right)(x^2+y^2)^{-3/2}2y$$

$$= \frac{-x^2y}{(x^2+y^2)^{3/2}}.$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{h^2}{\sqrt{h^2+0^2}} - 0\right)}{h} = \lim_{h\to 0} \frac{h}{|h|}.$$

Since $\lim\limits_{\substack{h\to 0\\ h>0}} \frac{h}{|h|} = \lim\limits_{\substack{h\to 0\\ h>0}} \frac{h}{h} = \lim\limits_{\substack{h\to 0\\ h>0}} 1 = 1$ and

$$\lim_{\substack{h\to 0\\ h<0}} \frac{h}{|h|} = \lim_{\substack{h\to 0\\ h<0}} \frac{h}{-h} = \lim_{\substack{h\to 0\\ h<0}} -1 = -1$$

then $\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \lim\limits_{h\to 0} \frac{h}{|h|}$ does not exist.

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0^2}{\sqrt{0^2+h^2}} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

So

$$\frac{\partial f}{\partial x} = \begin{cases} \dfrac{x^3+2xy^2}{(x^2+y^2)^{3/2}}, & \text{for } (x,y)\neq(0,0), \\ \text{does not exist}, & \text{for } (x,y)=(0,0) \end{cases}$$

and $$\frac{\partial f}{\partial y} = \begin{cases} \frac{-x^2y}{(x^2+y^2)^{3/2}}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Since x^3+2xy^2, $-x^2y$ and x^2+y^2 are polynomials and $x^2+y^2>0$ for $(x,y)\neq(0,0)$ then

x^3+2xy^2, $-x^2y$, $(x^2+y^2)^{3/2}$ are continuous and $(x^2+y^2)^{3/2} \neq 0$ for $(x,y)\neq(0,0)$,

and so

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous for $(x,y)\neq(0,0)$.

So f is C^1 for $(x,y)\neq(0,0)$ and so f is differentiable for $(x,y)\neq(0,0)$.

Since $\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)}$ does not exist

f is not differentiable for $(x,y)=(0,0)$

<u>§1.3 Example 3</u> Where is $f(x,y)=(x^2+y^2)^{3/4}$ C^1?

<u>Solution</u>: First compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x}=\frac{3}{4}(x^2+y^2)^{-1/4}2x=\frac{\frac{3}{2}x}{(x^2+y^2)^{1/4}} \quad \text{for } (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial y}=\frac{3}{4}(x^2+y^2)^{-1/4}2y=\frac{\frac{3}{2}y}{(x^2+y^2)^{1/4}} \quad \text{for } (x,y)\neq(0,0)$$

and

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)}=\lim_{h\to 0}\frac{(h^2)^{3/4}-0}{h}=\lim_{h\to 0}h^{1/2}=0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)}=\lim_{h\to 0}\frac{(h^2)^{3/4}-0}{h}=\lim_{h\to 0}h^{1/2}=0$$

So

$$\frac{\partial f}{\partial x}=\begin{cases}\frac{\frac{3}{2}x}{(x^2+y^2)^{1/4}}, & \text{for } (x,y)\neq(0,0)\\ 0, & \text{for } (x,y)=(0,0)\end{cases} \quad \text{and}$$

$$\frac{\partial f}{\partial y}=\begin{cases}\frac{\frac{3}{2}y}{(x^2+y^2)^{1/4}}, & \text{for } (x,y)\neq(0,0)\\ 0, & \text{for } (x,y)=(0,0)\end{cases}$$

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for $(x,y)\in\mathbb{R}^2$

Since $\frac{3}{2}x$, x^2+y^2 and $(x^2+y^2)^{1/4}$ are continuous for $(x,y)\neq 0$ and $(x^2+y^2)^{1/4}\neq 0$ then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous for $(x,y)\neq(0,0)$.

Using $x = r\cos\theta$ and $y = r\sin\theta$,

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} \frac{\frac{3}{2}x}{(x^2+y^2)^{1/4}}$$

$$= \lim_{r\to 0} \frac{\frac{3}{2} r\cos\theta}{(r^2)^{1/4}} = \lim_{r\to 0} \frac{\frac{3}{2} r\cos\theta}{r^{1/2}}$$

$$= \lim_{r\to 0} \frac{3}{2} r^{1/2} \cos\theta = 0 = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)}$$

and

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y} = \lim_{(x,y)\to(0,0)} \frac{\frac{3}{2}y}{(x^2+y^2)^{1/4}}$$

$$= \lim_{r\to 0} \frac{\frac{3}{2} r\sin\theta}{(r^2)^{1/4}} = \lim_{r\to 0} \frac{\frac{3}{2} r\sin\theta}{r^{1/2}}$$

$$= \lim_{r\to 0} \frac{3}{2} r^{1/2} \sin\theta = 0 = \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)}$$

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(x,y)=(0,0)$.

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous for $(x,y) \in \mathbb{R}^2$.

So f is C^1 for $(x,y) \in \mathbb{R}^2$.