Vector Calculus lecture 5

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§1.4 Example 5 Let $f: \mathbb{R}^3 \to \mathbb{R}^1$ and $g: \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(u,v,w) = u^2+v^2-w \quad \text{and} \quad g(x,y,z) = (x^2y,\, y^2,\, e^{-xz})$$

Compute $\underline{\underline{D}}(f\circ g)$ at $(x,y,z) = (0,1,2)$.

Solution:

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^1$$

$$(0,1,2) \mapsto (0,1,e^0) \longmapsto (0^2+1^2-1) = 0.$$

$$\underline{\underline{D}}f = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \end{pmatrix} = (2u \;\; 2v \;\; -1)$$

$$\underline{\underline{D}}g = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2y & 0 \\ -ze^{-xz} & 0 & -xe^{-xz} \end{pmatrix}$$

So

$$\underline{\underline{D}}(f\circ g)\Big|_{(x,y,z)=(0,1,2)} = \left(\underline{\underline{D}}f\Big|_{(u,v,w)=(0,1,1)}\right)\left(\underline{\underline{D}}g\Big|_{(x,y,z)=(0,1,2)}\right)$$

$$= \left((2u \;\; 2v \;\; -1)\Big|_{(u,v,w)=(0,1,1)}\right)\left(\begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2y & 0 \\ -ze^{-xz} & 0 & -xe^{-xz} \end{pmatrix}\Bigg|_{(x,y,z)=(0,1,2)}\right)$$

$$= (0 \;\; 2 \;\; -1)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix} = (2 \;\; 4 \;\; 0)$$

<u>§1.4 Example 6</u> Let

$f: \mathbb{R}^2 \to \mathbb{R}^3$ and $g: \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$f(x,y) = (x^2, 2x+y, y^3)$ and $g(u,v,w) = (u^2+2w, u-v^2)$.

Find $D(f \circ g \circ f)$ at $(x,y) = (1,0)$.

<u>Solution:</u>

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$$

$$(1,0) \longmapsto (1,2,0) \longmapsto (1,-3) \longmapsto (1,-4,9)$$

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 2 & 1 \\ 0 & 3y^2 \end{pmatrix}$$

$$Dg = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 2u & 0 & 2 \\ 1 & -2v & 0 \end{pmatrix}$$

$$D(f \circ g \circ f)\Big|_{(x,y)=(1,0)} = \left(Df\Big|_{(x,y)=(1,-3)}\right)\left(Dg\Big|_{(u,v,w)=(1,2,0)}\right)\left(Df\Big|_{(x,y)=(1,0)}\right)$$

$$= \begin{pmatrix} 2x & 0 \\ 2 & 1 \\ 0 & 3y^2 \end{pmatrix}\Bigg|_{(x,y)=(1,-3)} \cdot \begin{pmatrix} 2u & 0 & 2 \\ 1 & -2v & 0 \end{pmatrix}\Bigg|_{(u,v,w)=(1,2,0)} \cdot \begin{pmatrix} 2x & 0 \\ 2 & 1 \\ 0 & 3y^2 \end{pmatrix}\Bigg|_{(x,y)=(1,0)}$$

$$= \begin{pmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 27 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 4 \\ 5 & -4 & 4 \\ 27 & -4\cdot 27 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 2 & -4 \\ -6\cdot 27 & -4\cdot 27 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 2 & -4 \\ -162 & -108 \end{pmatrix}$$