03.08.2018
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Vector calculus Lecture 6

Taylor polynomials Let $f: \mathbb{R}^2 \to \mathbb{R}^1$.

The 0th order Taylor polynomial for f at (a_1, a_2) is

$$f(x_1, x_2) \approx f(a_1, a_2),$$

and there exists $\xi \in \mathbb{R}$ with $0 < \xi < 1$ such that

$$\text{Error} = \frac{1}{1!}\left(\left.\frac{\partial f}{\partial x_1}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} (x_1 - a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} (x_2 - a_2) \right)$$

where $(b_1, b_2) = (a_1 + \xi(x_1 - a_1), a_2 + \xi(x_2 - a_2))$.

The 1st order Taylor polynomial for f at (a_1, a_2) is

$$f(x_1, x_2) \approx f(a_1, a_2) + \frac{1}{1!}\left(\left.\frac{\partial f}{\partial x_1}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} \cdot (x_1 - a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} (x_2 - a_2) \right)$$

and there exists $\xi \in \mathbb{R}$ with $0 < \xi < 1$ such that

$$\text{Error} = \frac{1}{2!}\left(\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} \cdot (x_1 - a_1)^2 + 2 \left.\frac{\partial^2 f}{\partial x_1 \partial x_2}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} (x_1 - a_1)(x_2 - a_2) + \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} (x_2 - a_2)^2 \right)$$

where $(b_1, b_2) = (a_1 + \xi(x_1 - a_1), a_2 + \xi(x_2 - a_2))$.

The 2^{nd} order Taylor polynomial for f at (a_1, a_2) is

$$f(x_1,x_2) \approx f(a_1,a_2) + \frac{1}{1!}\left(\left.\frac{\partial f}{\partial x_1}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} \cdot (x_1 - a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} (x_2 - a_2) \right)$$

$$+ \frac{1}{2!}\left(\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} \cdot (x_1-a_1)^2 + 2 \left.\frac{\partial^2 f}{\partial x_1 \partial x_2}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} \cdot (x_1-a_1)(x_2-a_2) + \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{\substack{(x_1,x_2)\\=(a_1,a_2)}} (x_2-a_2)^2 \right)$$

and there exists $\xi \in \mathbb{R}$ with $0 < \xi < 1$ such that

$$\text{Error} = \frac{1}{3!}\left(\left.\frac{\partial^3 f}{\partial x_1^3}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} \cdot (x_1-a_1)^3 + 3 \left.\frac{\partial^3 f}{\partial x_1^2 \partial x_2}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} \cdot (x_1-a_1)^2(x_2-a_2) + 3 \left.\frac{\partial^3 f}{\partial x_1 \partial x_2^2}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} \cdot (x_1-a_1)(x_2-a_2)^2 + \left.\frac{\partial^3 f}{\partial x_2^3}\right|_{\substack{(x_1,x_2)\\=(b_1,b_2)}} (x_2-a_2)^3 \right)$$

where $(b_1, b_2) = (a_1 + \xi(x_1 - a_1), a_2 + \xi(x_2 - a_2))$.

§1.5 Example Find the second order Taylor polynomial for

$$f(x,y) = e^{xy} \quad \text{near } (1,0).$$

Hence approximate $e^{0.11}$

Solution:

$$f(x,y) \approx f(1,0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}\Big|_{(x,y)=(1,0)} \cdot (x-1) + \frac{\partial f}{\partial y}\Big|_{(x,y)=(1,0)} \cdot y \right)$$

$$+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(1,0)} (x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Big|_{(x,y)=(1,0)} (x-1)y + \frac{\partial^2 f}{\partial y^2}\Big|_{(x,y)=(1,0)} y^2 \right)$$

Then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{xy}) = y\,e^{xy} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^{xy}) = x\,e^{xy},$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = xy e^{xy} + e^{xy}, \quad \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}.$$

So

$$\frac{\partial f}{\partial x}\Big|_{(x,y)=(1,0)} = 0 \cdot e^0 = 0, \quad \frac{\partial f}{\partial y}\Big|_{(x,y)=(1,0)} = 1 \cdot e^0 = 1, \quad \frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(1,0)} = 0^2 e^0 = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}\Big|_{(x,y)=(1,0)} = 1 \cdot 0 e^0 + e^0 = 0 + 1 = 1, \quad \frac{\partial^2 f}{\partial y^2}\Big|_{(x,y)=(1,0)} = 1^2 \cdot e^0 = 1 \cdot 1 = 1.$$

and $f(1,0) = e^{1 \cdot 0} = e^0 = 1.$

So

$$f(x,y) \simeq 1 + \frac{1}{1!}\left(0\cdot(x-1) + 1\cdot y\right)$$

$$+ \frac{1}{2!}\left(0\cdot(x-1)^2 + 2\cdot 1\cdot(x-1)y + 1\cdot y^2\right)$$

$$= 1 + y + \tfrac{1}{2}\left(2(x-1)y + y^2\right)$$

$$= 1 + y + (x-1)y + \tfrac{1}{2}y^2 = 1 + xy + \tfrac{1}{2}y^2.$$

So

$$e^{0.11} = f(1.1, 0.1) \simeq 1 + (1.1)(0.1) + \tfrac{1}{2}(0.1)^2$$

$$= 1 + 0.11 + \tfrac{1}{2}(0.01) = 1.11 + 0.005$$

$$= 1.115.$$

An even closer approximation is 1.116278

§1.5 Example 2 Find a 2^{nd} order Taylor polynomial for

$$f(x,y) = \sin(x+2y) \text{ near } (0,0).$$

Find an upper bound for the error if

$$|x| < 0.1 \text{ and } |y| < 0.1.$$

Solution:

$$f(x,y) \approx f(0,0) + \frac{1}{1!}\left(\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} \cdot x + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} \cdot y \right)$$
$$+ \frac{1}{2!}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(0,0)} \cdot x^2 + 2 \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x,y)=(0,0)} xy + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x,y)=(0,0)} y^2 \right)$$

Now,

$$f(0,0) = \sin(0+2\cdot 0) = \sin 0 = 0.$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \left.\cos(x+2y)\right|_{(x,y)=(0,0)} = \cos(0+2 0) = 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = \left.2\cos(x+2y)\right|_{(x,y)=(0,0)} = 2\cos(0+2\cdot 0) = 2\cdot 1 = 2.$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+2y), \qquad \frac{\partial^2 f}{\partial y^2} = -4\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2\sin(x+2y).$$

So

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(0,0)} = -\sin(0+2\cdot 0) = -\sin 0 = 0$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x,y)=(0,0)} = -4\sin 0 = 0$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x,y)=(0,0)} = -2\sin 0 = 0.$$

So

$$f(x,y) \approx 0 + \frac{1}{1!}(1\cdot x + 2\cdot y) + \frac{1}{2!}(0\cdot x^2 + 2\cdot 0\cdot xy + 0\cdot y^2)$$

$$= x + 2y$$

$$\text{Error} = \frac{1}{3!}\left(\left.\frac{\partial^3 f}{\partial x^3}\right|_{(x,y)=(\xi_x,\xi_y)} x^3 + 3\left.\frac{\partial^3 f}{\partial x^2 \partial y}\right|_{(x,y)=(\xi_x,\xi_y)} x^2 y + 3\left.\frac{\partial^3 f}{\partial x \partial y^2}\right|_{(x,y)=(\xi_x,\xi_y)} xy^2 + \left.\frac{\partial^3 f}{\partial y^3}\right|_{(x,y)=(\xi_x,\xi_y)} y^3\right)$$

$$\frac{\partial^3 f}{\partial x^3} = -\cos(x+2y) \qquad \frac{\partial^3 f}{\partial x \partial y^2} = -4\cos(x+2y)$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = -2\cos(x+2y) \qquad \frac{\partial^3 f}{\partial y^3} = -8\cos(x+2y).$$

So

$$\text{Error} = \frac{1}{3!}\Big(-\cos(\xi x+2\xi y)x^3 + 3(-2\cos(\xi x+2\xi y))x^2y + 3(-4\cos(\xi x+2\xi y))xy^2 + (-8\cos(\xi x+2\xi y))y^3\Big)$$

$$= -\frac{1}{6}\cos(\xi x+2\xi y)\big(x^3+3x^2y(+2)+12xy^2+8y^3\big)$$

Since $0<\xi<1$ and $|x|<0.1$ and $|y|<0.1$

then

$$|\text{Error}| < \left|\frac{-1}{6}\cdot 1\,\big((0.1)^3+6(0.1)^3+12(0.1)^3+8(0.1)^3\big)\right|$$

$$= \left|-\frac{1}{6}\cdot 27\cdot(0.1)^3\right| = \frac{9}{2}(0.1)^3 = 4.5\cdot .001$$

$$= .0045.$$