Vector Calculus Lecture 8

Let $f: \mathbb{R}^2 \to \mathbb{R}$. The Hessian matrix of f is

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Let (a,b) be a critical point of f.

- If
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(a,b)} \in \mathbb{R}_{>0} \quad \text{and} \quad \left.\det(Hf)\right|_{(x,y)=(a,b)} \in \mathbb{R}_{>0}$$
then (a,b) is a maximum.

- If
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(a,b)} \in \mathbb{R}_{<0} \quad \text{and} \quad \left.\det(Hf)\right|_{(x,y)=(a,b)} \in \mathbb{R}_{>0}$$
then (a,b) is a minimum.

- If $\left.\det(Hf)\right|_{(x,y)=(a,b)} \in \mathbb{R}_{<0}$ then (a,b) is a saddle point.

- If $\left.\det(Hf)\right|_{(x,y)=(a,b)} = 0$ then (a,b) could be a maximum, could be a minimum, could be a saddle point, you don't know from this computation

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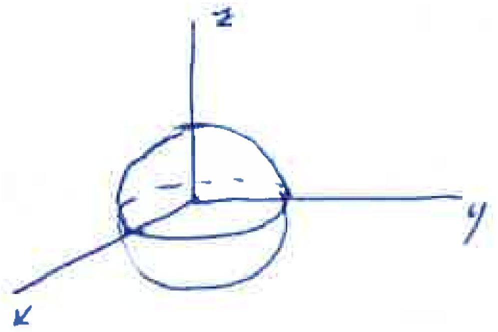
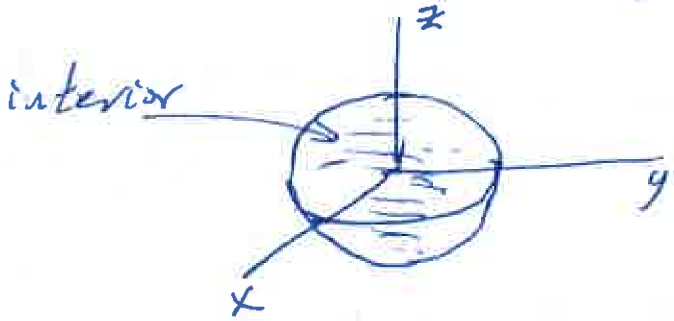
§1.7 Example 1: Graph $x^2+y^2+z^2=1$



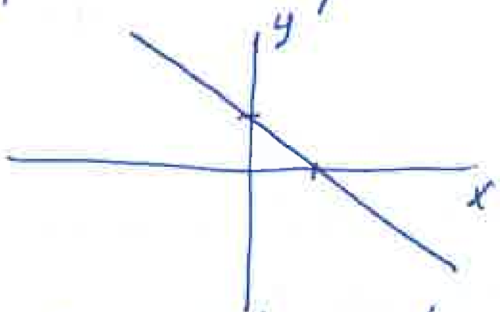
For a closed and bounded constraint there exists a minimum and a maximum.

§1.7 Example 2: Graph $x^2+y^2+z^2<1$.



For an open and bounded constraint maxima and minima need not exist. (but they might exist).

§1.7 Example 3: Graph $x+y=1$.



For an unbounded constraint maxima and minima need not exist (but they might).

§1.7 Example 5 Find the extrema of

$$f(x,y,z) = x+y+z$$

subject to the constraints

$$x^2+y^2=2 \quad \text{and} \quad x+z=1.$$

Solution: Critical points are when

$$\vec{\nabla} f = \lambda_1 \vec{\nabla} g_1 + \lambda_2 \vec{\nabla} g_2, \text{ which is}$$

$$\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}\right) = \lambda_1 \left(\frac{\partial g_1}{\partial x}\ \frac{\partial g_1}{\partial y}\ \frac{\partial g_1}{\partial z}\right) + \lambda_2 \left(\frac{\partial g_2}{\partial x}\ \frac{\partial g_2}{\partial y}\ \frac{\partial g_2}{\partial z}\right)$$

which is

$$(1,1,1) = \lambda_1(2x, 2y, 0) + \lambda_2(1,0,1)$$

since $g_1 = x^2+y^2-2$ and $g_2 = x+z-1$.

So critical points occur when

$$\begin{matrix} 2\lambda_1 x + \lambda_2 = 1 \\ 2\lambda_1 y + 0 = 1 \\ \lambda_2 = 1. \end{matrix} \quad \text{so} \quad \begin{matrix} 2\lambda_1 x + 1 = 1 \\ 2\lambda_1 y = 1. \end{matrix} \quad \text{so} \quad \begin{matrix} 2\lambda_1 x = 0 \\ 2\lambda_1 y = 1. \end{matrix}$$

Since $2\lambda_1 y = 1$ then $\lambda_1 \neq 0$.

Since $\lambda_1 \neq 0$ and $2\lambda_1 x = 0$ then $x = 0$.

Since $x^2+y^2=2$ then $y^2=2$ and $y = \pm\sqrt{2}$.

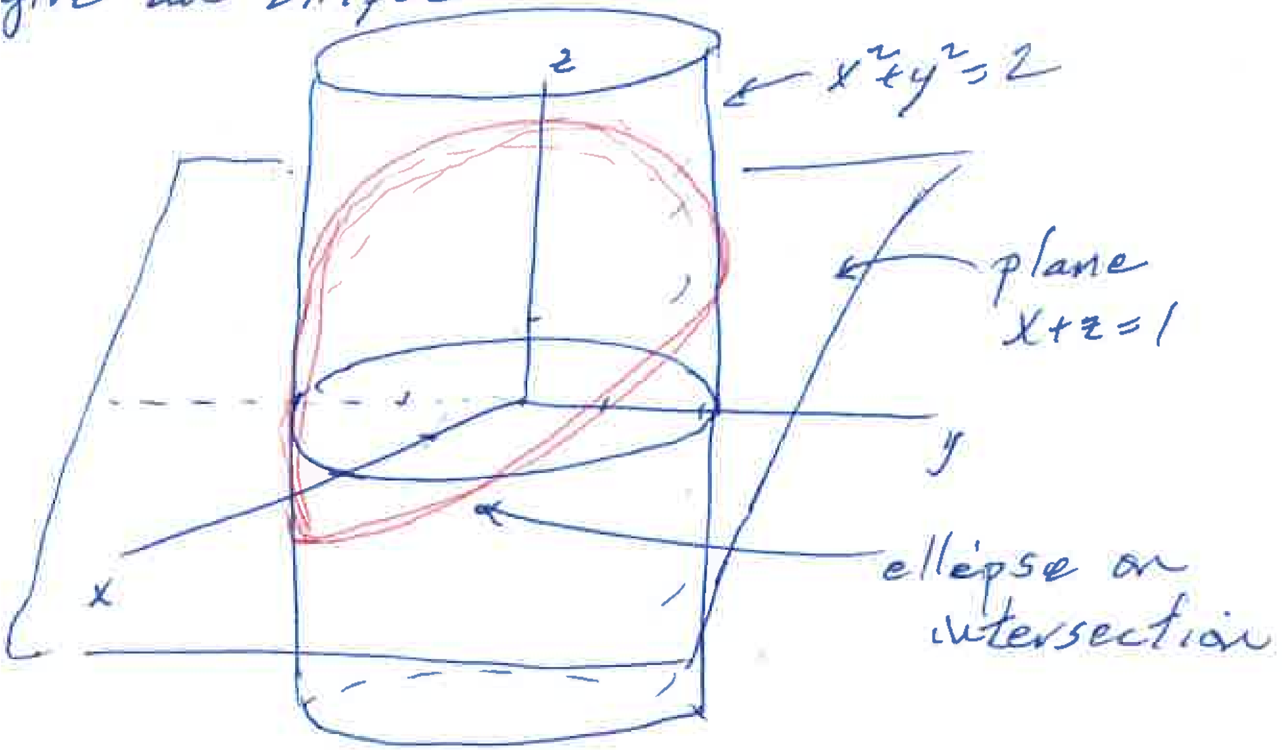
Since $x+z=1$ then $z=1$.

So the critical points are

$$(0, \sqrt{2}, 1) \quad \text{and} \quad (0, -\sqrt{2}, 1).$$

The constraints $x^2+y^2=2$ and $x+z=1$ A. Ram

give an ellipse



Since this is a closed bounded constraint there must be a minimum and a maximum of f. Since

$$f(0,\sqrt{2},1) = 0+\sqrt{2}+1 = 1+\sqrt{2}$$

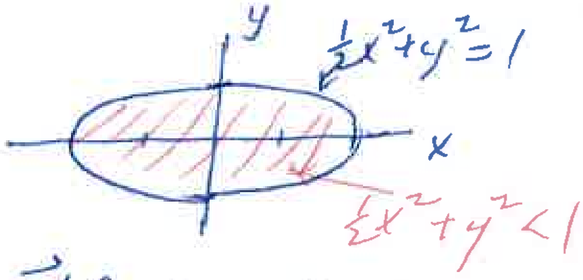
$$f(0,-\sqrt{2},1) = 0-\sqrt{2}+1 = 1-\sqrt{2}$$

then $1+\sqrt{2} = f(0,\sqrt{2},1)$ is the maximum of f and $1-\sqrt{2} = f(0,-\sqrt{2},1)$ is the minimum of f subject to the constraints.

§1.7 Example 6 Find the absolute maximum and absolute minimum of f where

$$f(x,y) = \tfrac{1}{2}(x^2+y^2)$$

in the region $\tfrac{1}{2}x^2+y^2 \le 1$.



Solution:

Part 1: Inside: $\tfrac{1}{2}x^2+y^2<1$

Critical points are when $\vec{\nabla} f = 0$ which is

$$\left(\tfrac{1}{2}2x, \tfrac{1}{2}2y\right) = (0,0)$$

So critical points are at $x=0, y=0$.

Then $$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(0,0)} = \left.\frac{\partial}{\partial x}(x)\right|_{(x,y)=(0,0)} = 1\Big|_{(x,y)=(0,0)} = 1 \in \mathbb{R}_{>0}$$

and
$$\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\Bigg|_{(x,y)=(0,0)} = 1\Big|_{(x,y)=(0,0)}$$

$$= 1 \in \mathbb{R}_{>0}.$$

So $(x,y)=(0,0)$ has $f(0,0)=0$ is a minimum of f in the region $x^2+y^2<1$.

Part 2: ~~The~~ boundary $\frac{1}{4}x^2+y^2=1$.

The constraint is $g=\frac{1}{4}x^2+y^2-1$.

The critical points are when

$\vec{\nabla} f = \lambda \vec{\nabla} g$, which is

$(x,y) = \lambda(\frac{1}{4}2x, 2y)$, which is

$x = \frac{1}{2}\lambda x$ and $y = 2\lambda y$.

So $x(1-\lambda)=0$ and $y(1-2\lambda)=0$.

So $x=0$ or $\lambda=1$ and $y=0$ or $\lambda=\frac{1}{2}$.

Since $\frac{1}{4}x^2+y^2=1$ we don't have $(x,y)=(0,0)$.

We can't have both $\lambda=1$ and $\lambda=\frac{1}{2}$ so

either $x=0$ or $y=0$.

Since $\frac{1}{4}x^2+y^2=1$ the critical points are

$(2,0)$ and $(-2,0)$ and $(0,1)$ and $(0,-1)$.

with

$f(2,0)=\frac{1}{2}\cdot(2^2+0^2)=2$, $f(0,1)=\frac{1}{2}(0^2+1^2)=\frac{1}{2}$

$f(-2,0)=\frac{1}{2}((-2)^2+0^2)=2$, $f(0,-1)=\frac{1}{2}(0^2+(-1)^2)=\frac{1}{2}$.

So $(0,1)$ and $(0,-1)$ are where the minima of f occur and $(2,0)$ and $(-2,0)$ are where the maxima of f occur.

The constraint $\frac{1}{4}x^2+y^2=1$ is a closed and bounded constraint.