Q1 ①

(1) $$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(a) For fixed x with $x \neq 0$ then

$2xy$ is a polynomial in y,

x^2+y^2 is a polynomial in y,

$x^2+y^2 \neq 0$,

and so $\dfrac{2xy}{x^2+y^2}$ is a continuous function of y.

For fixed x with $x = 0$ then

$$f(0,y) = \begin{cases} \dfrac{0}{0+y^2}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases} = 0,$$

which is a constant, and continuous, function.

~~(b) For fixed y with $y \neq 0$ then~~

~~$2xy$ is a polynomial in x,~~

~~x^2+y^2 is a polynomial in x,~~

~~$x^2+y^2 \neq 0$,~~

~~and so $\dfrac{2xy}{x^2+y^2}$ is a continuous function of y.~~

~~For fixed y with $y = 0$ then $f(x,0) = 0$,~~

~~which is a constant, and continuous, function.~~

(b) Since

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{y\to 0}\frac{0}{0^2+y^2} = \lim_{y\to 0} 0 = 0, \text{ and}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \lim_{x\to 0}\frac{2x^2}{x^2+x^2} = \lim_{x\to 0} 1 = 1$$

then $f(x,y)$ is not continuous at $(0,0)$.

(c) If $x=0$ then $f(0,y)=0$ and

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = 0, \text{ so } \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} \text{ exists.}$$

If $y=0$ then $f(x,0)=0$ and

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = 0, \text{ so } \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} \text{ exists.}$$

If $(a,b)\neq(0,0)$ then

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)} = \left.\left(\frac{2y}{x^2+y^2} + \frac{2xy(-1)2x}{(x^2+y^2)^2}\right)\right|_{(x,y)=(a,b)}$$

$$= \left.\frac{2yx^2+2y^3-4x^2y}{(x^2+y^2)^2}\right|_{(x,y)=(a,b)}$$

$$= \frac{2b^3-2a^2b}{(a^2+b^2)^2}.$$

$$\lim_{\substack{(a,b)\to(0,0)\\ b=2a}} \frac{\partial f}{\partial x} = \lim_{a\to 0}\frac{2\cdot 8a^3-4a^3}{(a^2+4a^2)^2} = \lim_{a\to 0}\frac{12a^3}{25a^4} = \lim_{a\to 0}\frac{12}{25a}$$

Q1 ③

does not exist. So $\frac{\partial f}{\partial x}$ is not continuous at $(a,b)=(0,0)$.

Similarly $\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)} = \frac{2a^3-2b^2a}{(a^2+b^2)^2}$

and $\lim_{\substack{(a,b)\to(0,0)\\ a=2b}} \frac{\partial f}{\partial y} = \lim_{b\to 0} \frac{12}{25b}$ does not exist.

So $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$.

(2) $f(x,y)=xy$ and $g(x,y)=\frac{1}{8}x^2+\frac{1}{2}y^2-1$.

Then $\vec{\nabla} f=\lambda\cdot\vec{\nabla} g$ gives

$(y,x)=\lambda\left(\frac{1}{4}x,y\right)$ so that $y=\frac{1}{4}\lambda x$ and $x=\lambda y$.

So $y=\frac{1}{4}\lambda\cdot\lambda y=\frac{1}{4}\lambda^2 y$.

So $(1-\frac{1}{4}\lambda^2)y=0$. So $y=0$ or $\lambda=\pm 2$.

Case 1: $y=0$. Then $x=\lambda y$ gives $x=0$.

But $(x,y)=(0,0)$ does not satisfy the constraint

$\frac{1}{8}x^2+\frac{1}{2}y^2=1$,

so this case cannot occur.

Case 2: $\lambda=\pm 2$. Then $y=\pm\frac{1}{2}x$ and $x=\pm 2y$.

The constraint gives

$$0=\frac{1}{8}x^2+\frac{1}{2}y^2-1=\frac{1}{8}(\pm 2y)^2+\frac{1}{2}y^2-1=\frac{1}{2}y^2+\frac{1}{2}y^2-1$$
$$=y^2-1.$$

So $y=\pm 1$. So the critical points are

$(2,1)$, $(-2,1)$, $(2,-1)$ and $(-2,-1)$

since $x=\pm 2y$.

The constraint $\frac{1}{8}x^2+\frac{1}{2}y^2=1$ (an ellipse) is closed and bounded, so

$(2,1)$ and $(-2,-1)$ with $f(2,1)=f(-2,-1)=2$

are maxima and $(-2,1)$ and $(2,-1)$ with $f(-2,1)=f(2,-1)=-2$ are minima.

(3) Since $c(t) = (2(-t+\sin t), \sqrt{2}(1-\cos t), \sqrt{2}(1-\cos t))$ then

$$\frac{dc}{dt} = (2(-1+\cos t), \sqrt{2}\sin t, \sqrt{2}\sin t)$$

Since $\sin t = \sin(2\cdot\frac{t}{2}) = 2\sin\frac{t}{2}\cos\frac{t}{2}$

$$\cos t = \cos(2\cdot\frac{t}{2}) = \cos^2\frac{t}{2} - \sin^2\frac{t}{2} = 1 - 2\sin^2\frac{t}{2}$$

then $-1+\cos t = -2\sin^2\frac{t}{2}$ and

$$\frac{dc}{dt} = (-2\cdot 2\sin^2\frac{t}{2}, \sqrt{2}\cdot 2\sin\frac{t}{2}\cos\frac{t}{2}, \sqrt{2}\, 2\sin\frac{t}{2}\cos\frac{t}{2})$$

$$= 2\sin\frac{t}{2}(-2\sin\frac{t}{2}, \sqrt{2}\cos\frac{t}{2}, \sqrt{2}\cos\frac{t}{2})$$

Thus

$$\frac{ds}{dt} = \left|\frac{d\vec{c}}{dt}\right| = (4\sin^2\frac{t}{2}(4\sin^2\frac{t}{2} + 2\cos^2\frac{t}{2} + 2\cos^2\frac{t}{2}))^{\frac{1}{2}}$$

$$= 2\sin\frac{t}{2}(4(\sin^2\frac{t}{2} + \cos^2\frac{t}{2}))^{\frac{1}{2}} = 2\sin\frac{t}{2}\cdot 4^{\frac{1}{2}}$$

$= 4\sin\frac{t}{2}$ (which is positive for $0 < t < 2\pi$).

(a) The arclength along c for $0 < t < 2\pi$ is

$$s = \int_0^{2\pi} \frac{ds}{dt}\, dt = \int_0^{2\pi} 4\sin(\tfrac{t}{2})\, dt = -4\cos(\tfrac{t}{2})\cdot 2\Big|_0^{2\pi}$$

$$= -8(\cos\pi - \cos 0) = -8(-1-1) = 16.$$

(b) $\vec{T}(t) = \dfrac{\frac{dc}{dt}}{\left|\frac{dc}{dt}\right|} = \dfrac{1}{4\sin\frac{t}{2}}\left(2\sin\frac{t}{2}\left(-2\sin\frac{t}{2}, \sqrt{2}\cos\frac{t}{2}, \sqrt{2}\cos\frac{t}{2}\right)\right)$

$= \frac{1}{2}\left(-2\sin\frac{t}{2}, \sqrt{2}\cos\frac{t}{2}, \sqrt{2}\cos\frac{t}{2}\right)$

$= \left(-\sin\frac{t}{2}, \dfrac{\sqrt{2}\cos\left(\frac{t}{2}\right)}{2}, \dfrac{\sqrt{2}\cos\frac{t}{2}}{2}\right)$

So $\dfrac{d\vec{T}}{dt} = \left(-\frac{1}{2}\cos\left(\frac{t}{2}\right), -\frac{\sqrt{2}}{4}\sin\left(\frac{t}{2}\right), -\frac{\sqrt{2}}{4}\sin\left(\frac{t}{2}\right)\right)$ and

$\left|\dfrac{d\vec{T}}{dt}\right| = \left(\frac{1}{4}\cos^2\left(\frac{t}{2}\right) + \frac{2}{16}\sin^2\left(\frac{t}{2}\right) + \frac{2}{16}\sin^2\left(\frac{t}{2}\right)\right)^{1/2}$

$= \left(\frac{1}{4}\cos^2\frac{t}{2} + \frac{1}{4}\sin^2\frac{t}{2}\right)^{1/2} = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}.$

So $\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = 2\left(-\frac{1}{2}\cos\frac{t}{2}, -\frac{\sqrt{2}}{4}\cancel{\sin}\,\frac{t}{2}, -\frac{\sqrt{2}}{4}\sin\frac{t}{2}\right)$

$= \left(-\cos\frac{t}{2}, -\frac{\sqrt{2}}{2}\sin\frac{t}{2}, -\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right)$

Then

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\frac{t}{2} & \frac{\sqrt{2}}{2}\cos\frac{t}{2} & \frac{\sqrt{2}}{2}\cos\frac{t}{2} \\ -\cos\frac{t}{2} & -\frac{\sqrt{2}}{2}\sin\frac{t}{2} & -\frac{\sqrt{2}}{2}\sin\frac{t}{2} \end{vmatrix}$

$= \hat{\imath}\left(-\frac{1}{2}\sin\frac{t}{2}\cos\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2}\cos\frac{t}{2}\right)$

$-\hat{\jmath}\left(\frac{\sqrt{2}}{2}\sin^2\frac{t}{2} + \frac{\sqrt{2}}{2}\cos^2\frac{t}{2}\right) + \hat{k}\left(\frac{\sqrt{2}}{2}\sin^2\frac{t}{2} + \frac{\sqrt{2}}{2}\cos^2\frac{t}{2}\right)$

$= \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$

(c) $\kappa(t) = \dfrac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \dfrac{\frac{1}{2}}{4\sin\frac{t}{2}} = \dfrac{1}{8\sin\left(\frac{t}{2}\right)}$

Since $\vec{B}(t) = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ then

$$\frac{d\vec{B}}{ds} = \frac{\frac{d\vec{B}}{dt}}{\left|\frac{ds}{dt}\right|} = \frac{\vec{0}}{4\sin\frac{t}{2}} = \vec{0} \quad \text{(since } B \text{ is a constant)}$$

$= (0, 0, 0)$.

Since $\tau(t)$ is such that $\frac{d\vec{B}}{ds} = -\tau(t)\vec{N}(t)$

then $\tau(t) = 0$.

This indicates that the curve lies in a plane, the plane $y = z$.

This is the plane $0x + y - z = 0$.

(4)(a) $$\vec{\nabla}(e^f) = \left(\frac{\partial e^f}{\partial x}, \frac{\partial e^f}{\partial y}, \frac{\partial e^f}{\partial z}\right)$$

$$= \left(e^f\frac{\partial f}{\partial x}, e^f\frac{\partial f}{\partial y}, e^f\frac{\partial f}{\partial z}\right)$$

$$= e^f\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = e^f(\vec{\nabla} f)$$

(b) $\nabla^2 e^f = \vec{\nabla}\cdot(\vec{\nabla} e^f) = \vec{\nabla}\cdot(e^f\vec{\nabla} f)$, by part (a).

Using $\vec{\nabla}\cdot(f\vec{F}) = f(\vec{\nabla}\cdot\vec{F}) + \vec{F}\cdot\vec{\nabla} f$,

$$\nabla^2 e^f = \vec{\nabla}\cdot(e^f\vec{\nabla} f) = e^f(\vec{\nabla}\cdot\vec{\nabla} f) + \vec{\nabla} f\cdot\vec{\nabla} e^f$$

$$= e^f(\nabla^2 f) + \vec{\nabla} f\cdot e^f\vec{\nabla} f$$

$$= e^f(\nabla^2 f + \vec{\nabla} f\cdot\vec{\nabla} f)$$

(c) Using $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\vec{\nabla} f\cdot\vec{\nabla} g$,

$$\nabla^2(ge^h - he^g) = g\nabla^2 e^h + e^h\nabla^2 g + 2\vec{\nabla} g\cdot\vec{\nabla} e^h$$

$$- h\nabla^2 e^g - e^g\nabla^2 h + 2\vec{\nabla} h\cdot\vec{\nabla} e^g$$

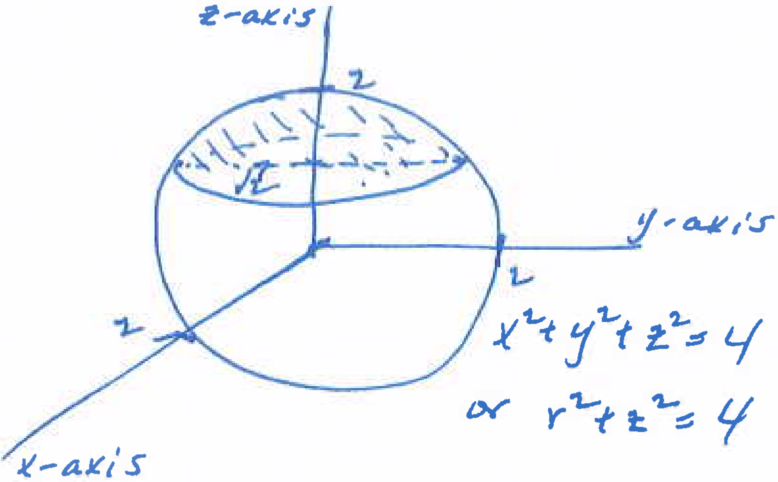
$$= ge^h(\nabla^2 h + \vec{\nabla} h\cdot\vec{\nabla} h) + e^h\nabla^2 g + 2\vec{\nabla} g\cdot e^h\vec{\nabla} h$$

$$- he^g(\nabla^2 g + \vec{\nabla} g\cdot\vec{\nabla} g) - e^g\nabla^2 h + 2\vec{\nabla} h\cdot e^g\vec{\nabla} g$$

by parts (a) and (b). Then using $\nabla^2 g = \nabla^2 h = \vec{\nabla} g\cdot\vec{\nabla} h = 0$,

$$\nabla^2(ge^h - he^g) = ge^h(\vec{\nabla} h\cdot\vec{\nabla} h) - he^g(\vec{\nabla} g\cdot\vec{\nabla} g).$$

(5)



$x^2+y^2+z^2=4$

or $r^2+z^2=4$

(a)

$$\iiint_D dx\,dy\,dz = \int_{z=\sqrt{2}}^{z=2}\int_{\theta=0}^{2\pi}\int_{r=0}^{r=(4-z^2)^{\frac{1}{2}}} r\,dr\,d\theta\,dz$$

$$= \int_{z=\sqrt{2}}^{z=2}\int_{\theta=0}^{\theta=2\pi}\left(\tfrac{1}{2}r^2\Big|_{r=0}^{r=(4-z^2)^{\frac{1}{2}}}\right)d\theta\,dz$$

$$= \int_{z=\sqrt{2}}^{z=2}\int_{\theta=0}^{\theta=2\pi}\tfrac{1}{2}(4-z^2)\,d\theta\,dz$$

$$= \int_{z=\sqrt{2}}^{z=2}\left(\tfrac{1}{2}(4-z^2)\,\theta\Big|_{\theta=0}^{\theta=2\pi}\right)dz$$

$$= \int_{z=\sqrt{2}}^{z=2}\pi(4-z^2)\,dz = 4\pi z - \frac{\pi}{3}z^3\Big|_{z=\sqrt{2}}^{z=2}$$

$$= \left(8\pi - \frac{8}{3}\pi\right) - \left(4\pi\sqrt{2} - \frac{\pi}{3}2\sqrt{2}\right)$$

$$= \frac{16}{3}\pi - \frac{10\sqrt{2}}{3}\pi = \frac{(16-10\sqrt{2})}{3}\pi$$

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(b) The sphere of radius 2 is parametrized by

$$\Phi(\varphi,\theta) = (2\sin\theta\cos\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\theta)$$

Then
$$\vec{T}_\varphi = (-2\sin\theta\sin\varphi,\ 2\sin\theta\cos\varphi,\ 0)$$
$$\vec{T}_\theta = (2\cos\theta\cos\varphi,\ 2\cos\theta\sin\varphi,\ -2\sin\theta)$$

$$\vec{T}_\varphi \times \vec{T}_\theta = (-4\sin^2\theta\cos\varphi,\ -4\sin^2\theta\sin\varphi,\ -4\sin\theta\cos\theta)$$

$$|\vec{T}_\varphi \times \vec{T}_\theta| = \sqrt{4^2\sin^4\theta\cos^2\varphi + 4^2\sin^4\theta\sin^2\varphi + 4^2\sin^2\theta\cos^2\theta}$$
$$= 4\sqrt{\sin^4\theta + \sin^2\theta\cos^2\theta} = 4\sin\theta$$

So the surface area is

$$\iint_S dS = \iint_S |\vec{T}_\varphi \times \vec{T}_\theta|\, d\theta\, d\varphi = \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\frac{\pi}{4}} 4\sin\theta\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\left(-4\cos\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{4}}\right)d\varphi = \int_{\varphi=0}^{\varphi=2\pi}\left(-4\cdot\frac{\sqrt{2}}{2} - (-4\cdot 1)\right)d\varphi$$

$$= (-2\sqrt{2}+4)\,\varphi\Big|_{\varphi=0}^{\varphi=2\pi} = 2\pi(2-\sqrt{2})\cdot 2 = 4\pi(2-\sqrt{2}).$$