Calculus I Lecture 14

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$\mathbb{R}^3$ is the set of triples of real numbers,

$$\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$$

Addition: $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Scalar multiplication $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$

Example 2.25 Let $\vec{v} = (3, -2)$ and $\vec{w} = (1, 5)$.
Calculate $\vec{v} + \vec{w}$, $\vec{v} - \vec{w}$, $\vec{w} - \vec{v}$, $3\vec{v}$.

Solution:

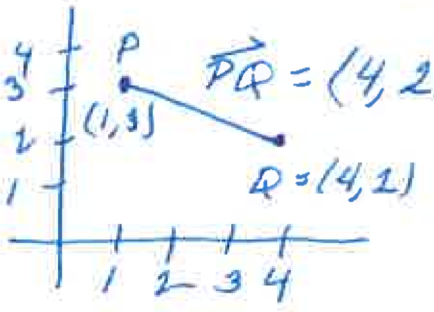
$$\vec{v} + \vec{w} = (3, -2) + (1, 5) = (3+1, -2+5) = (4, 3),$$

$$\vec{v} - \vec{w} = (3, -2) - (1, 5) = (3, -2) + (-1)(1, 5)$$
$$= (3, -2) + (-1, -5) = (3-1, -2-5) = (2, -7),$$

$$\vec{w} - \vec{v} = (1, 5) - (3, -2) = (1, 5) + (-3, 2) = (1-3, 5+2)$$
$$= (-2, 7),$$

$$3\vec{v} = 3(3, -2) = (9, -6).$$

Example 2.26 Graphing



$\vec{PQ} = (4, 2) - (1, 3) = (3, -1)$

The <u>length</u> of a vector $\vec{v} = (a_1, a_2, a_3)$ in $\mathbb{R}^3$ is

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

<u>Example 2.30</u> Find the length of $(1,2)$ and $(-1,3,-2)$

<u>Solution</u>: $\|(1,2)\| = \sqrt{1^2+2^2} = \sqrt{5}$.

$$\|(-1,3,-2)\| = \sqrt{(-1)^2+3^2+(-2)^2} = \sqrt{1+9+4} = \sqrt{14} \quad //$$

<u>Example 2.31</u> Prove that $\|c\vec{u}\| = |c|\,\|\vec{u}\|$.

<u>Solution</u> Let $\vec{u} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

To show: $\|c\vec{u}\| = |c|\,\|\vec{u}\|$.

$$\|c\vec{u}\| = \|c(a_1, a_2, \dots, a_n)\| = \|(ca_1, ca_2, \dots, ca_n)\|$$

$$= \sqrt{(ca_1)^2 + (ca_2)^2 + \dots + (ca_n)^2}$$

$$= \sqrt{c^2a_1^2 + c^2a_2^2 + \dots + c^2a_n^2}$$

$$= \sqrt{c^2(a_1^2 + a_2^2 + \dots + a_n^2)}$$

$$= |c|\sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = |c|\,\|\vec{u}\|. \quad //$$

<u>Example 2.31</u> Find the distance between $P = (-2,1,0)$ and $Q = (3,-1,1)$.

<u>Solution</u>: distance between P and Q

$$= \|Q-P\| = \|(3,-1,1) - (-2,1,0)\| = \|(5,-2,1)\|$$

$$= \sqrt{25+4+1} = \sqrt{30} \quad //$$

A <u>unit vector</u> is a vector of length 1.

If $\vec{u}$ is a vector then

$$\frac{1}{\|\vec{u}\|}\vec{u} \text{ is a unit vector}$$

and so is $\frac{-1}{\|\vec{u}\|}\vec{u}$.

<u>Example 2.33</u> Let $\vec{v}$ be the vector from $A=(2,0,-1)$ to $B=(1,2,-3)$.

Find two unit vectors that are parallel to $\vec{v}$.

<u>Solution:</u>

$$\vec{v} = B - A = (1,2,-3)-(2,0,-1)$$

$$= (1-2,\ 2-0,\ -3+1) = (-1,2,-2)$$

Then

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{1^2+2^2+2^2}}(-1,2,-2) = \frac{1}{\sqrt{9}}(-1,2,-2)$$

$$= \frac{1}{3}(-1,2,-2) = \left(\frac{-1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

and

$$\frac{-1}{\|\vec{v}\|}\vec{v} = -\left(\frac{-1}{3}, \frac{2}{3}, -\frac{2}{3}\right) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

are unit vectors parallel to v. $\square$

Example 2.34 Let $\hat{i}=(1,0)$ and $\hat{j}=(0,1)$.
Express $\vec{v}=(1,2)$ and $\vec{u}=(-1,3)$ in terms of $\hat{i}$ and $\hat{j}$.

Solution: $\vec{v}=(1,2)=(1,0)+(0,2)$
$$= \hat{i}+2(0,1)=\hat{i}+2\hat{j}.$$
$$\vec{u}=(-1,3)=(-1,0)+(0,3)$$
$$=(-1)(1,0)+3(0,1)=(-1)\hat{i}+3\hat{j} \,//$$

The scalar product of $\vec{u}$ and $\vec{v}$ is
$$\vec{u}\cdot\vec{v}=u_1v_1+\cdots+u_nv_n \quad \text{if } \vec{u}=(u_1,\ldots,u_n)$$
$$\text{and } \vec{v}=(v_1,\ldots,v_n)$$

Example 2.35 Let $\vec{u}=(2,3,-1)$ and $\vec{v}=(4,5,0)$
Calculate $\vec{u}\cdot\vec{v}$.

Solution $\vec{u}\cdot\vec{v}=(2,3,-1)\cdot(4,5,0)$
$$=2\cdot4+3\cdot5+(-1)\cdot0$$
$$=8+15+0=23 \,//.$$

<u>Example 2.37</u> Let $\vec{u} = (a_1, a_2, \dots a_n)$ and $\vec{v} = (b_1, \dots, b_n)$ and let $c \in \mathbb{R}$.

(a) Prove that $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$

(b) Prove that $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$.

<u>Solution</u> (b) $\vec{u}\cdot\vec{u} = (a_1, a_2, \dots, a_n)\cdot(a_1, a_2, \dots, a_n)$

$$= a_1^2 + a_2^2 + \cdots + a_n^2 = \left(\sqrt{a_1^2 + \cdots + a_n^2}\right)^2$$

$$= \|\vec{u}\|^2.$$

(a) $c(\vec{u}\cdot\vec{v}) = c((a_1, a_2, \dots, a_n)\cdot(b_1, b_2, \dots, b_n))$

$$= c(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)$$

$$= c a_1 b_1 + c a_2 b_2 + \cdots + c a_n b_n.$$

and

$c\vec{u}\cdot\vec{v} = (c(a_1, a_2, \dots, a_n))\cdot\vec{v}$

$$= (ca_1, ca_2, \dots, ca_n)\cdot(b_1, b_2, \dots, b_n)$$

$$= ca_1 b_1 + ca_2 b_2 + \cdots + ca_n b_n.$$

and

$\vec{u}\cdot c\vec{v} = (a_1, a_2, \dots, a_n)\cdot(c(b_1, b_2, \dots, b_n))$

$$= (a_1, a_2, \dots, a_n)\cdot(cb_1, cb_2, \dots, cb_n)$$

$$= a_1 c b_1 + a_2 c b_2 + \cdots + a_n c b_n$$

$$= ca_1 b_1 + ca_2 b_2 + \cdots + ca_n b_n.$$

So $c(\vec{u}\cdot\vec{v}) = c\vec{u}\cdot\vec{v} = \vec{u}\cdot c\vec{v}$. $\square$