Calculus I: Parametric curves

Example 2.44 Find the cartesian equation of the curve given by

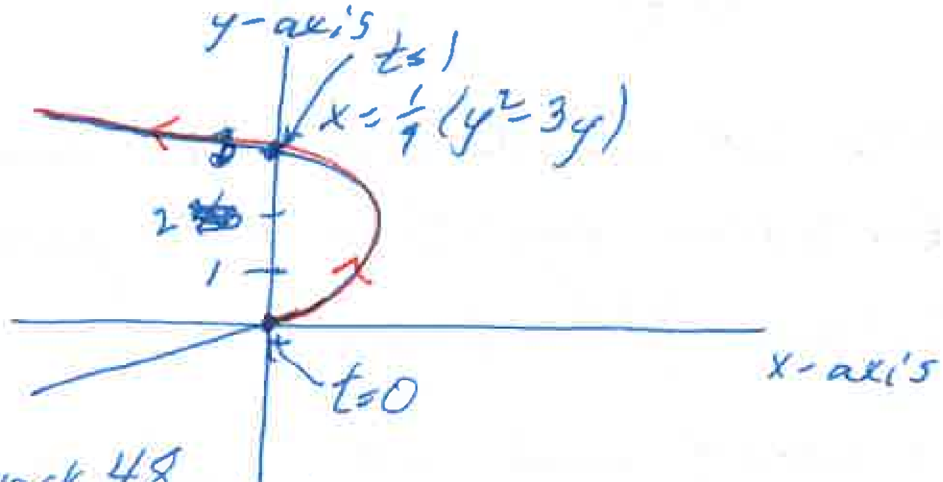
$$\vec{r}(t) = (t^2-t)\hat{\imath} + 3t\,\hat{\jmath} \quad \text{for} \quad t \in \mathbb{R}_{\geq 0}.$$

Graph the path.

Solution: $y = 3t$ and $x = t^2 - t = \frac{(3t)^2}{9} - \frac{(3t)}{3}$

So $x = \frac{y^2}{9} - \frac{y}{3} = \frac{1}{9}(y^2 - 3y)$.

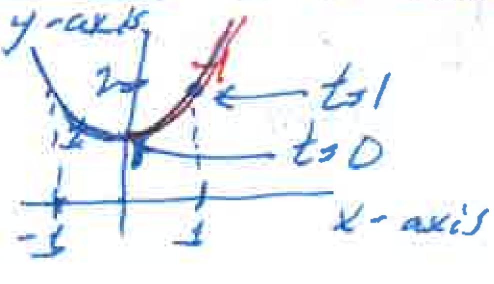
So $9x = y^2 - 3y$.



Homework 48
~~Example 2.45~~ Find the Cartesian equation of the path of the particle with position given by

$$\vec{r}(t) = t\,\hat{\imath} + (t^2+1)\,\hat{\jmath} \quad \text{for} \quad t \in \mathbb{R}_{\geq 0}.$$

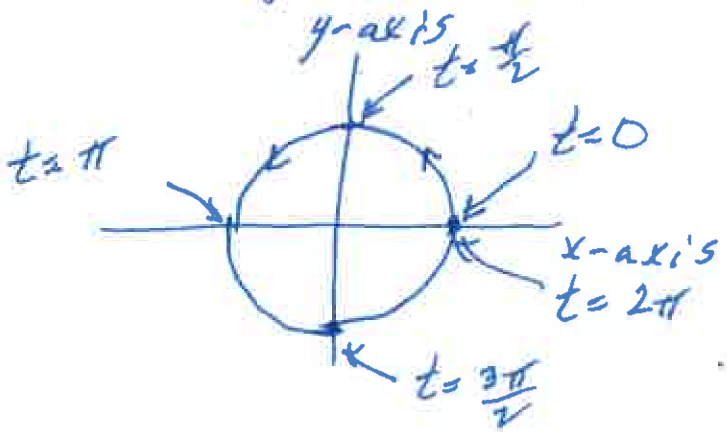
Solution $x = t$ and $y = t^2+1 = x^2+1$



Example 2.45 Find the Cartesian equation of the path of a particle with position given by $\vec{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j}$ for $t \in \mathbb{R}$.

Solution $x = \cos t$ and $y = \sin t$ and

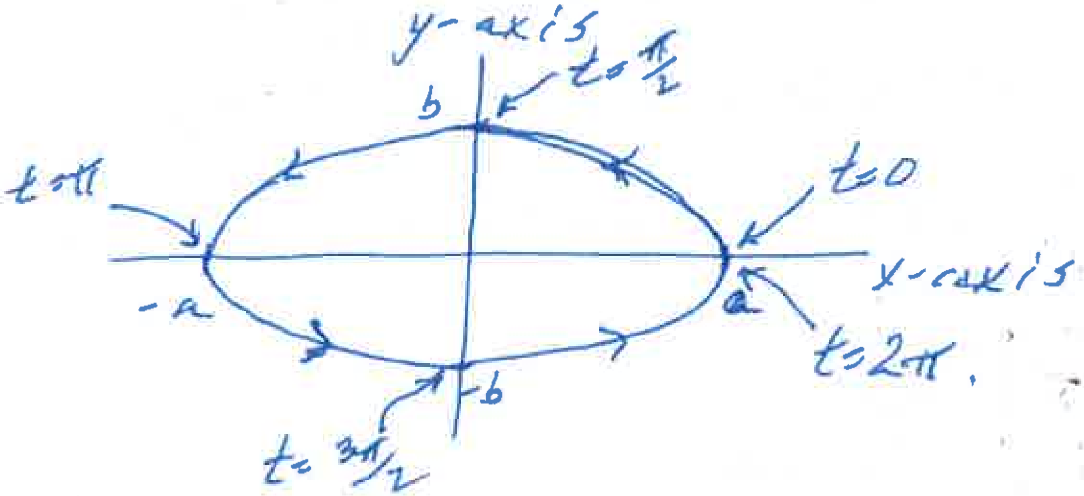
$$x^2 + y^2 = 1.$$



Example 2.46 Let $a, b \in \mathbb{R}$ with $a \neq 0$ and $b \neq 0$. Find the Cartesian equation of the parametric equation $\vec{r}(t) = a\cos(t)\, \hat{i} + b\sin t\, \hat{j}$.

Solution $x = a\cos t$ and $y = b\sin t$ and

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

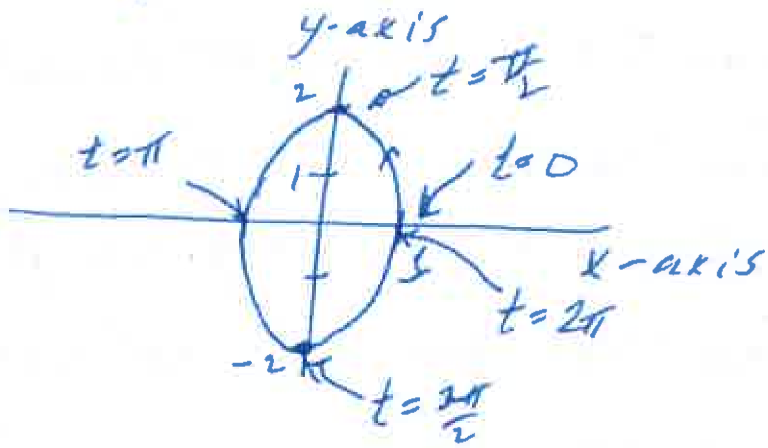


<u>Example 2.47</u> Find the Cartesian equation of the path of a particle with position given by

$$\vec{r}(t) = \cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} \text{ for } t \in \mathbb{R}.$$

<u>Solution:</u> $x = \cos t$ and $y = 2\sin t$ and

$$x^2 + \left(\frac{y}{2}\right)^2 = 1.$$



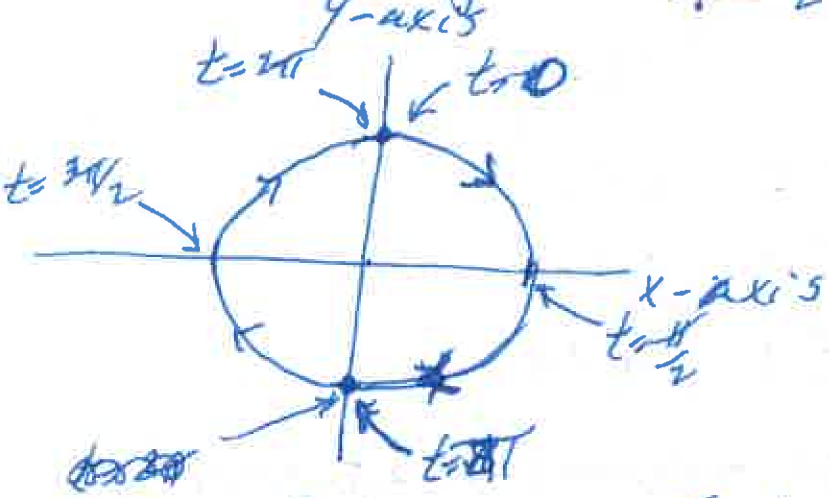
<u>Example 2.48</u> Find the Cartesian equation of the parametric equation

$$\vec{r} = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \text{ for } t \in \mathbb{R}.$$

<u>Solution:</u> $x = \sin t$ and $y = \cos t$ and

$$x^2 + y^2 = 1.$$

$$\vec{r}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \cancel{= \cos(t-\tfrac{\pi}{2})\,\hat{\imath} + \sin(t-\tfrac{\pi}{2})}$$



$$\vec{r}(t) = \cos\left(\tfrac{\pi}{2} - t\right)\hat{\imath} + \sin\left(\tfrac{\pi}{2} - t\right)\hat{\jmath}.$$

Example 2.49 Find the Cartesian equation and graph the parametric equation for $\vec{r}: \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ given by

$$\vec{r}(t) = \cos(2t)\,\hat{\imath} - 2\sin(2t)\,\hat{\jmath}.$$

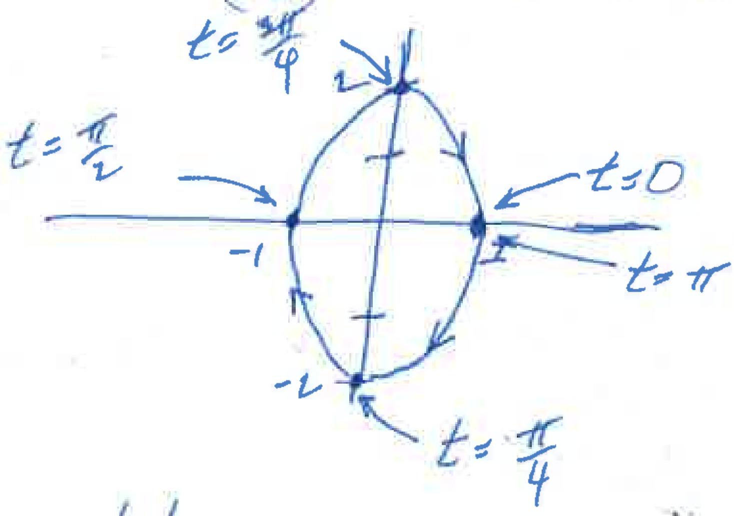
Find the position at times $t=0$, $t=\frac{\pi}{4}$ and $t=\frac{\pi}{2}$.

Find the time taken for the particle to return to its original position.

Find the direction of motion.

Solution: $x = \cos(2t)$ and $y = -2\sin(2t)$

and $x^2 + \left(\frac{y}{-2}\right)^2 = 1$ so that $x^2 + \frac{y^2}{4} = 1$.



Position at $t=0$: $(1,0)$

Position at $t=\frac{\pi}{4}$: $(0,-2)$

Position at $t=\frac{\pi}{2}$: $(-1,0)$

Time to return to original position: π

Direction of motion: counterclockwise.

Example 2.50 The motion of two particles is given by the equations

$\vec{r}_1(t) = (t+1)\hat{i} + (t^2-4t)\hat{j}$ and

$\vec{r}_2(t) = 2t\hat{i} + (6t-9)\hat{j}$.

Determine

(a) the times and points at which the points collide

(b) the distance between the particles at $t=9$.

Solution (a) If $\vec{r}_1(t) = \vec{r}_2(t)$ then

$(t+1)\hat{i} + (t^2-4t)\hat{j} = 2t\hat{i} + (6t-9)\hat{j}$

and $t+1=2t$ and $t^2-4t = 6t-9$.

So $t=1$ and $1^2-4 = 6-9$. (OK).

So the points collide at $t=1$ with position

$\vec{r}_1(1) = (1+1)\hat{i} + (1^2-4)\hat{j} = 2\hat{i} - 3\hat{j} = (2,-3)$

(b) At $t=9$, $\vec{r}_1(9) = (9+1)\hat{i} + (9^2-4\cdot 9)\hat{j} = (10, 55)$

and $\vec{r}_2(9) = 18\hat{i} + (54-9)\hat{j} = (18, 45)$

The distance between these points is

$|(10,55)-(18,45)| = |(-8,10)| = \sqrt{64+100} = \sqrt{164}$

$= 2\sqrt{41}$.