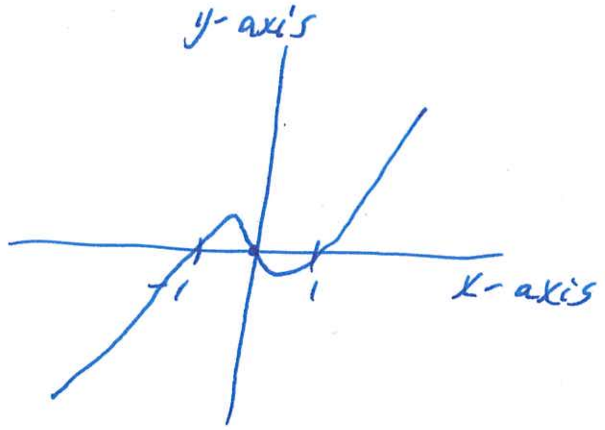
Example 3.15 Find the global maximum and minimum of

$f: [-1, \sqrt{3}] \to \mathbb{R}$ given by $f(x) = \sqrt{3}(x^3 - x)$.

Solution $f(x) = \sqrt{3}(x^3 - x) = \sqrt{3}x(x^2-1) = \sqrt{3}x(x+1)(x-1)$

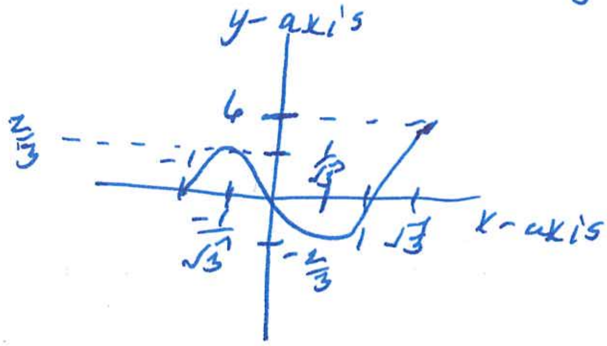


Notes: (a) As $x \to \infty$, $f(x) \to \infty$

(b) As $x \to -\infty$ then $f(x) \to -\infty$

(c) $f(0) = 0$, $f(1) = 0$, $f(-1) = 0$

For $f: [-1, \sqrt{3}] \to \mathbb{R}$ the graph is



with

$f(\sqrt{3}) = \sqrt{3}((\sqrt{3})^3 - \sqrt{3})$

$= \sqrt{3}^4 - \sqrt{3}^2 = 9 - 3 = 6$

$\frac{df}{dx} = \sqrt{3}(3x^2-1) = \sqrt{3}\cdot 3(x^2 - \frac{1}{3}) = \sqrt{3}\cdot 3(x + \frac{1}{\sqrt{3}})(x - \frac{1}{\sqrt{3}})$

So $\frac{df}{dx} = 0$ if $x = \frac{-1}{\sqrt{3}}$ or $x = \frac{1}{\sqrt{3}}$.

$f(\frac{1}{\sqrt{3}}) = \sqrt{3}((\frac{1}{\sqrt{3}})^3 - \frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}^2} - 1 = \frac{1}{3} - 1 = -\frac{2}{3}$

$f(\frac{-1}{\sqrt{3}}) = \sqrt{3}((\frac{-1}{\sqrt{3}})^3 - \frac{-1}{\sqrt{3}}) = \frac{-1}{\sqrt{3}^2} + 1 = \frac{-1}{3} + 1 = \frac{2}{3}$

So $f(\sqrt{3}) = 6$ and $f(\frac{1}{\sqrt{3}}) = -\frac{2}{3}$ are the global maximum and global minimum.

<u>Example 3.18</u> The position of the rock is

$$f(t) = 40 - 4.9t^2 \text{ metres}$$

Find the velocity and the acceleration, and the velocity when the rock hits the ground.

<u>Solution</u>: velocity = change in position with time = $\frac{df}{dt}$.

acceleration = change in velocity with time = $\frac{d}{dt}\left(\frac{df}{dt}\right)$

$$\text{velocity} = \frac{df}{dt} = \frac{d(40 - 4.9t^2)}{dt} = 0 - 4.9\cdot 2t = -9.8t \text{ m/s}$$

$$\text{acceleration} = \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d}{dt}(-9.8t) = -9.8 \text{ m/s}^2$$

When the rock hits the ground $f(t) = 0$ so that

$$0 = 40 - 4.9t^2 = 4.9\left(\frac{40}{4.9} - t^2\right) = 4.9\left(\sqrt{\frac{40}{4.9}} - t\right)\left(\sqrt{\frac{40}{4.9}} + t\right)$$

so $t = \sqrt{\frac{40}{4.9}} = \sqrt{\frac{4\cdot 10}{49/10}} = \frac{2}{7}\sqrt{100} = \frac{2}{7}\cdot 10 = \frac{20}{7}$ seconds.

The velocity when $t = \frac{20}{7}$ is

$$-9.8\left(\frac{20}{7}\right) = -2\cdot 4.9\left(\frac{20}{.7}\right) = -2\cdot 7\cdot(.7)\left(\frac{20}{.7}\right)$$

$$= -14\cdot 20 = -280 \text{ m/s} \text{ (moving downwards)}.$$

<u>Example 3.21</u> Determine where

$$f: \mathbb{R} \to \mathbb{R} \qquad g: \mathbb{R} \to \mathbb{R} \qquad h: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^4 \qquad x \mapsto x^3 \qquad x \mapsto x^{1/3}$$

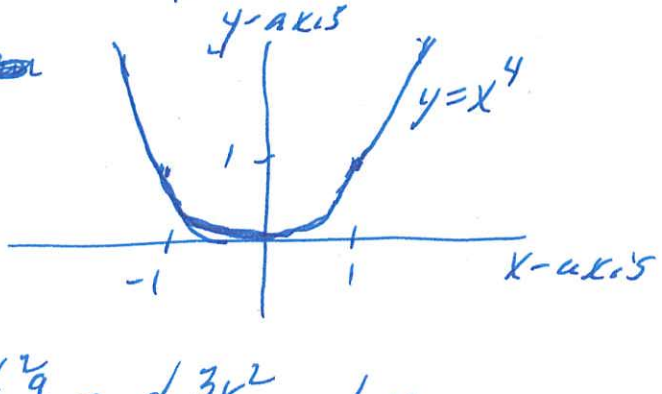
are concave up and concave down and determine any points of inflection.

<u>Solution:</u> $\frac{df}{dx} = \frac{d\,x^4}{dx} = 4x^3$ and $\frac{d^2f}{dx^2} = \frac{d\,4x^3}{dx} = 12x^2$

So $\frac{d^2f}{dx^2} > 0$ if $x \neq 0$.

So f is concave up for $x \neq 0$.

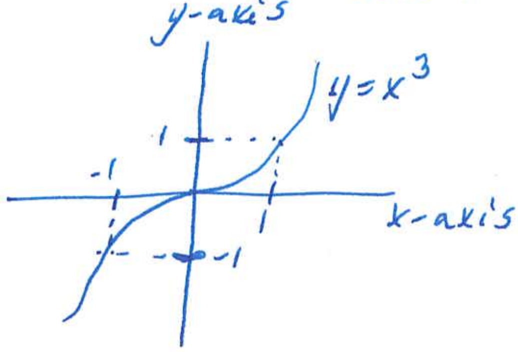
By drawing chords one can argue that $f(x) = x^4$ is also concave up at $x = 0$ (as $f(x) > 0$ for $x \neq 0$)

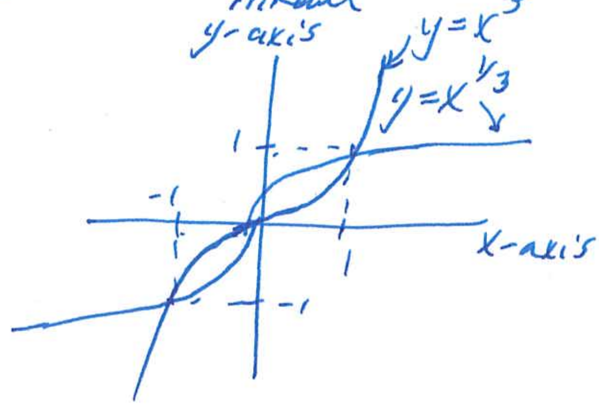


$\frac{dg}{dx} = \frac{d\,x^3}{dx} = 3x^2$ and $\frac{d^2g}{dx^2} = \frac{d\,3x^2}{dx} = 6x$.

So $\frac{d^2g}{dx^2} > 0$ if $x > 0$ and $\frac{d^2g}{dx^2} < 0$ if $x < 0$

So $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is concave up for $x > 0$ and concave down for $x < 0$ and $x = 0$ is a point of inflection.





$$\frac{dh}{dx} = \frac{d\, x^{1/3}}{dx} = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-2/3} \quad \text{and}$$

$$\frac{d^2h}{dx^2} = \frac{d\left(\frac{1}{3}x^{-2/3}\right)}{dx} = \frac{1}{3}\left(\frac{-2}{3}\right) x^{-2/3-1} = \frac{-2}{9} x^{-5/3}$$

If $x > 0$ then $\frac{d^2h}{dx^2} < 0$ and h is concave down.

If $x < 0$ then $\frac{d^2h}{dx^2} > 0$ and h is concave up

Then $x = 0$ is a point of inflection for h.

<u>Example 3.24</u> Write down the first five derivatives of $f(x) = \sin x$

<u>Solution</u> $\frac{d \sin x}{dx} = \cos x$,

$$\frac{d^2 \sin x}{dx^2} = \frac{d \cos x}{dx} = -\sin x,$$

$$\frac{d^3 (+\sin x)}{dx^3} = \frac{d(-\sin x)}{dx} = -\cos x,$$

$$\frac{d^4 (\sin x)}{dx^4} = \frac{d(-\cos x)}{dx} = -(-\sin x) = \sin x.$$

$$\frac{d^5(\sin x)}{dx} = \frac{d \sin x}{dx} = \cos x.$$

Example 3.25 Find $f^{(n)}$ for $n \leq 6$ if

$$f(x) = ax^3 + bx^2 + cx + d.$$

Solution

$$f^{(1)} = \frac{df}{dx} = \frac{d}{dx}(ax^3+bx^2+cx+d)$$
$$= a3x^2 + b\cdot 2x + c\cdot 1 + 0$$
$$= 3ax^2 + 2bx + c.$$

$$f^{(2)} = \frac{d^2f}{dx^2} = \frac{d(3ax^2+2bx+c)}{dx} = 3a\cdot 2x + 2b\cdot 1 + 0$$
$$= 6ax + 2b$$

$$f^{(3)} = \frac{d^3f}{dx^3} = \frac{d}{dx}(6ax+2b) = 6a\cdot 1 + 0 = 6a.$$

$$f^{(4)} = \frac{d^4f}{dx^4} = \frac{d}{dx}(6a) = 0.$$

$$f^{(5)} = \frac{d^5f}{dx^5} = \frac{d\cdot 0}{dx} = 0$$

$$f^{(6)} = \frac{d^6f}{dx^6} = \frac{d\,0}{dx} = 0.$$