Calculus I Lecture 26

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The curve $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ has a <u>cusp</u> at $t=a$ if

(1) $\vec{r}\,'(t) = 0$, and

(2) $\frac{dx}{dt}$ changes sign at $t=a$

or $\frac{dy}{dt}$ changes sign at $t=a$.

A curve $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ is <u>smooth</u> if it has no cusps.

<u>Integrals</u> (Three forms)

(1) Antiderivatives: $\int \frac{df}{dx}\,dx = f$

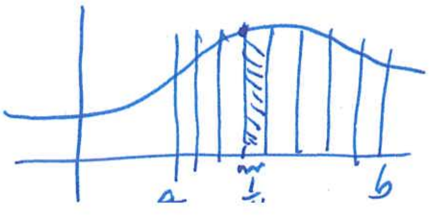
$\int \; dx$ undoes $\frac{d}{dx}$.

(2) Antiderivatives with endpoints:

$$\int \frac{df}{dx}\,dx \Big]_{x=a}^{x=b} = f(b) - f(a)$$

(3) Adding up slices:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty}\left(f(a)\tfrac{1}{N} + f\left(a+\tfrac{1}{N}\right)\tfrac{1}{N} + \cdots + f\left(b-\tfrac{2}{N}\right)\tfrac{1}{N} + f\left(b-\tfrac{1}{N}\right)\tfrac{1}{N}\right)$$



$= \lim_{N\to\infty}$ (sum of little rectangles of height given by f and width $1/N$).

Theorem (Fundamental Theorem of calculus)

$$\int_a^b g(x)\,dx = \int g\,dx \Big]_{x=a}^{x=b}.$$

Since antiderivaties are backwards of $\frac{d}{dx}$

(a) $\frac{dx}{dx} = 1$, gives $\int 1\,dx = x + c$, where c is a constant.

(b) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$, gives $\int (f+g)\,dx = \int f\,dx + \int g\,dx$

(c) $\frac{d(cf)}{dx} = c\frac{df}{dx}$, gives $\int cf\,dx = c\int f\,dx$, if c is a constant

(d) $\frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx}g$, gives $\int u\,dv = uv - \int v\,du$

(e) $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$, gives $\int f\frac{du}{dx}\,dx = \int f\,dx$

Example 4.3 $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}.$

Example 4.4 Let s, c be constants.

$$\frac{d}{dx}\left(\frac{1}{s}\arctan\left(\frac{x}{s}\right)+c\right) = \frac{1}{s}\,\frac{1}{1+\left(\frac{x}{s}\right)^2}\left(\frac{1}{s}\right)+0$$

$$= \frac{1}{s^2}\,\frac{1}{\frac{(s^2+x^2)}{s^2}} = \frac{1}{s^2+x^2}.$$

So $\int \frac{1}{s^2+x^2}\,dx = \frac{1}{s}\arctan\left(\frac{x}{s}\right)+c.$

Example 4.5 $\int \left(5e^{2x} - \frac{3}{x}\right)dx$

$= \frac{5}{2}e^{2x} - 3\log x + c$, where c is a constant,

because $\frac{d}{dx}\left(\frac{5}{2}e^{2x} - 3\log x + c\right) = \frac{5}{2}2e^{2x} - 3\frac{1}{x} + 0$

$$= 5e^{2x} - \frac{3}{x}.$$

Example 4.6 $\int_1^{e^{\pi/2}} \frac{\sin(\log x)}{x}\,dx = \int_{x=1}^{x=e^{\pi/2}} \sin(\log x)\frac{1}{x}\,dx$

$= -\cos(\log x)\Big]_{x=1}^{x=e^{\pi/2}} = -\cos(\log e^{\pi/2}) - (-\cos(\log 1))$

$= -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = 1.$

Example 4.7 $\int_{x=0}^{x=1} 2x e^{-x^2}\,dx = -e^{-x^2}\Big]_{x=0}^{x=1}$

$= -e^{-1^2} - (-e^{-0^2}) = -e^{-1} + e^{0} = -\frac{1}{e} + 1$

$= 1 - \frac{1}{e}$.

Example 4.8 $\int 2x(x^2-5)^4\,dx = \frac{1}{5}(x^2-5)^5 + c,$

where c is a constant.

Example 4.9 $\int \cos(3x)(\sin(3x)+4)^{\frac{1}{2}}\,dx$

$= \frac{1}{3}\cdot\frac{2}{3}(\sin(3x)+4)^{3/2} + c$

$= \frac{2}{9}(\sin(3x)+4)^{3/2} + c$, where c is a constant.

Check: $\frac{d}{dx}\left(\frac{2}{9}(\sin(3x)+4)^{3/2} + c\right)$

$= \frac{2}{9}\cdot\frac{3}{2}(\sin(3x)+4)^{\frac{1}{2}}(\cos(3x)\cdot 3 + 0) + 0$

$= \frac{1}{3}(\sin(3x)+4)^{\frac{1}{2}}\cos(3x)\cdot 3 = \cos(3x)(\sin(3x)+4)^{\frac{1}{2}}$.