MAST10005 Slide

A. Ram 21.07.2019

Sets

$s \in S$ means s is an element of the set S

ϕ is the **emptyset**, the set with no elements

$T \subseteq S$ means if $t \in T$ then $t \in S$. (T is a <u>subset</u> of S).

$T = S$ means $T \subseteq S$ and $S \subseteq T$

$S \cup T = \{u \mid u \in S \text{ or } u \in T\}$ (union)

$S \cap T = \{u \mid u \in S \text{ and } u \in T\}$ (intersection)

$S \times T = \{(s,t) \mid s \in S \text{ and } t \in T\}$ (product)

Calculus 1

Lecture 2
A. Ram

Example 1.1 Let $A = \{n \in \mathbb{Z}_{>0} \mid \sin n > 0\}$

and $B = \{n \in \mathbb{Z}_{>0} \mid \sin^2 n \le \sin n\}$.

Prove that $A \subseteq B$.

Solution: To show: $A \subseteq B$.

To show: If $x \in A$ then $x \in B$.

Assume $x \in A$.

Then $x \in \mathbb{Z}_{>0}$ and $\sin x > 0$.

To show: $x \in B$.

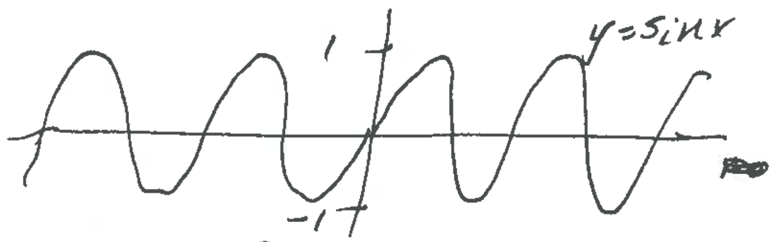
To show: $x \in \mathbb{Z}_{>0}$ and $\sin^2 x \le \sin x$.

To show: $\sin^2 x \le \sin x$.

Using that $\sin x > 0$ to show $\dfrac{\sin^2 x}{\sin x} \le \dfrac{\sin x}{\sin x}$.

To show: $\sin x \le 1$.

This is true because the graph of $y = \sin x$ is



so that if $x \in \mathbb{R}$ then $\sin x \le 1$.

So $\sin x \le 1$.

So $x \in B$.

So $A \subseteq B$. //

Example 1.8 Let $A = \{3n+1 \mid n \in \mathbb{Z}\}$ and $B = \{6m+1 \mid m \in \mathbb{Z}\}$. Show that $A \not\subseteq B$.

Solution: To show: $A \not\subseteq B$.

To show: There exists $x \in A$ such that $x \notin B$.

Let $x = 4$.

To show: $x \notin B$.

To show: There does not exist $m \in \mathbb{Z}$ such that $x = 6m+1$.

To show: (contrapositive) If $x = 6m+1$ then $m \notin \mathbb{Z}$.

Assume $x = 6m+1$.

Then $m = \frac{x-1}{6} = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$.

So $m \notin \mathbb{Z}$.

So $x \notin B$.

So $A \not\subseteq B$. //

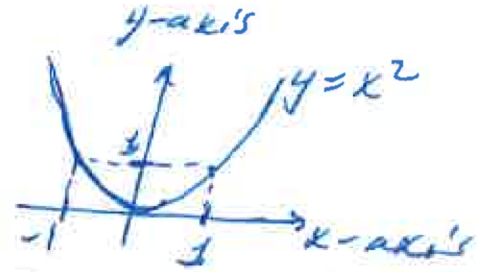
Example 1.9 Express the following sets as unions of intervals.

(a) $\{x \in \mathbb{R} \mid x^2 > 1\}$

(b) $\{x \in [-2\pi, 2\pi] \mid \sin x \leq 0\}$

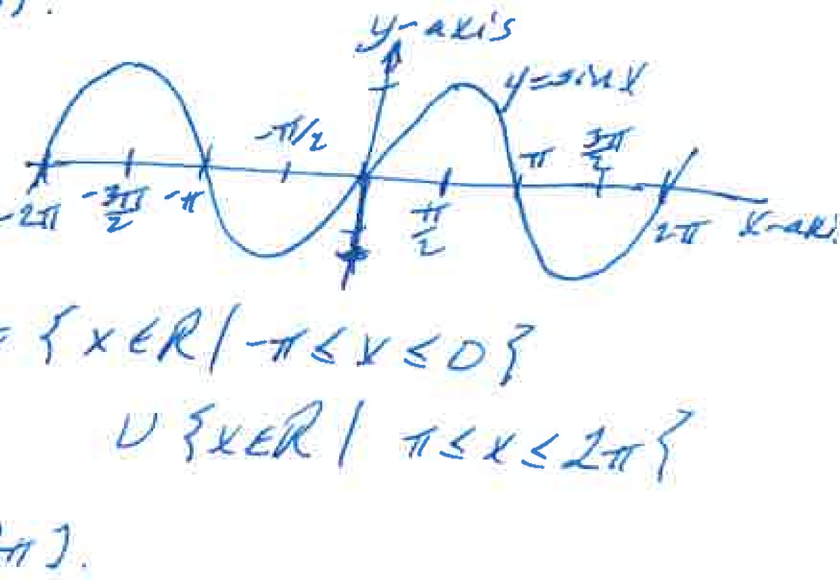
(c) $\{x \in [-2, 2] \mid x \notin \mathbb{Z}\}$

Solution (a) Using the graph



$$\{x \in \mathbb{R} \mid x^2 > 1\} = \{x \in \mathbb{R} \mid x < -1\} \cup \{x \in \mathbb{R} \mid x > 1\}$$
$$= \mathbb{R}_{(-\infty,-1)} \cup \mathbb{R}_{(1,\infty)}.$$

(b) Using the graph



$$\{x \in [-2\pi, 2\pi] \mid \sin x \leq 0\} = \{x \in \mathbb{R} \mid -\pi \leq x \leq 0\} \cup \{x \in \mathbb{R} \mid \pi \leq x \leq 2\pi\}$$
$$= \mathbb{R}_{[-\pi,0]} \cup \mathbb{R}_{[\pi,2\pi]}.$$

(c) $\{x \in [-2,2] \mid x \notin \mathbb{Z}\} = \{x \in [-2,2] \mid x \notin \{-2,-1,0,1,2\}\}$
$$= \mathbb{R}_{(-2,-1)} \cup \mathbb{R}_{(-1,0)} \cup \mathbb{R}_{(0,1)} \cup \mathbb{R}_{(1,2)}$$

Calculus I

Example 1.10

(a) Express $(2,8)\cup[3,10]$ as an interval.

(b) Is the set $(0,\sqrt{2}]\cup[\frac{\pi}{2},3)$ an interval?

Solution:

(a)

$$(2,8)\cup[3,10] = \{x\in\mathbb{R} \mid 2<x<8 \text{ or } 3\le x\le 10\}$$
$$= \{x\in\mathbb{R} \mid 2<x\le 10\} = \mathbb{R}_{(2,10]}.$$

(b)

Since $\sqrt{2}\approx 1.414$ and $\frac{\pi}{2}>1.5$ then

$1.47\notin(0,\sqrt{2}]\cup[\frac{\pi}{2},3)$ and $\sqrt{2}<1.47<\frac{\pi}{2}$.

So $(0,\sqrt{2}]\cup[\frac{\pi}{2},3)$ is not an interval.

Example 1.11

(a) Express $(2,8)\cap[3,10]$ as an interval.

(b) Express $(0,\sqrt{2}]\cap[\frac{\pi}{2},3)$ in the simplest possible way.

Solution (a) ~~As above (2,8)~~ From the graph above,

$$(2,8)\cap[3,10] = \{x\in\mathbb{R} \mid 2<x<8 \text{ and } 3\le x\le 10\}$$
$$= \{x\in\mathbb{R} \mid 3\le x<8\} = \mathbb{R}_{[3,8)}$$

(b) $(0,\sqrt{2}]\cap[\frac{\pi}{2},3) = \{x\in\mathbb{R} \mid 0<x\le\sqrt{2} \text{ and } \frac{\pi}{2}\le x<3\}$

$\cancel{\{x\in\mathbb{R} \mid}$ $=\emptyset$.

Example 1.11 (c) Express $\mathbb{Z} \cap [-\pi, \pi]$ in "list of elements" form.

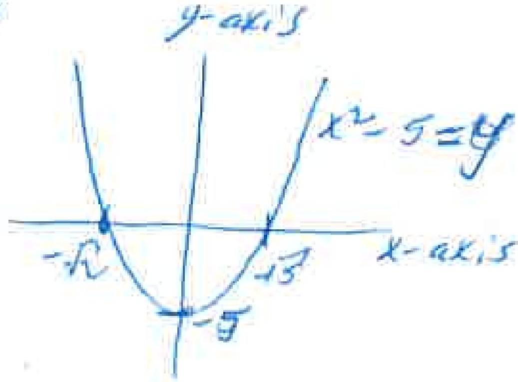
(d) Express $\mathbb{Z} \cap \{x \in \mathbb{R} \mid x^2 - 5 < 0\}$ in "list of elements" form.

Solution

(c) $\mathbb{Z} \cap [-\pi, \pi] = \{x \in \mathbb{Z} \mid -\pi \leq x \leq \pi\}$

$= \{x \in \mathbb{Z} \mid -3.14 \leq x \leq 3.14\}$

$= \{-3, -2, -1, 0, 1, 2, 3\}$



(d) $\mathbb{Z} \cap \{x \in \mathbb{R} \mid x^2 - 5 < 0\}$

$= \{x \in \mathbb{Z} \mid x^2 - 5 < 0\}$

$= \{x \in \mathbb{Z} \mid x^2 < 5\} = \{0, 1, -1, 2, -2\}$

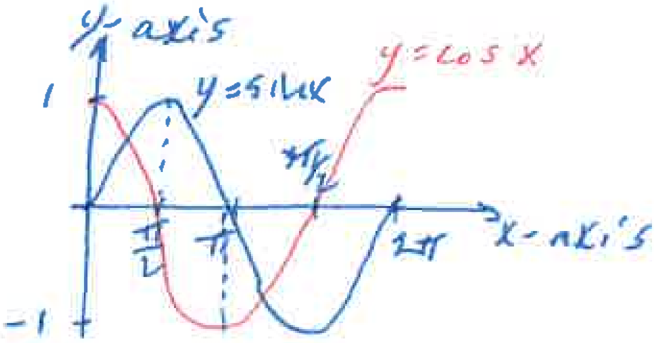
$= \{-2, -1, 0, 1, 2\}$

Example 1.12 Express the following as intersection and as single set descriptions.

(a) The set of reals with positive sine and negative cosine.

Solution $\{x\in\mathbb{R} \mid \sin x>0 \text{ and } \cos x<0\}$

$=\{x\in\mathbb{R}\mid \sin x>0\}\cap\{x\in\mathbb{R}\mid\cos x<0\}$



$=\{x+2k\pi \mid k\in\mathbb{Z},\ 0<x<\pi\}\cap\{x+2k\pi\mid k\in\mathbb{Z},\ \frac{\pi}{2}<x<\frac{3\pi}{2}\}$

$=\{x+2k\pi\mid k\in\mathbb{Z},\ \frac{\pi}{2}<x<\pi\} = \bigcup_{k\in\mathbb{Z}} \mathbb{R}_{(\frac{\pi}{2}+2k\pi,\ \pi+2k\pi)}$

$=\bigcup_{k\in\mathbb{Z}}\mathbb{R}_{((2k+\frac{1}{2})\pi,\ (2k+1)\pi)}$

(b) The set of integers with positive sine

Solution $\{x\in\mathbb{Z}\mid\sin x>0\} = \mathbb{Z}\cap\{x\in\mathbb{R}\mid \sin x>0\}$

$=\mathbb{Z}\cap\left(\bigcup_{k\in\mathbb{Z}}\mathbb{R}_{(0+2k\pi,\ \pi+2k\pi)}\right)$

$=\mathbb{Z}\cap\left(\bigcup_{k\in\mathbb{Z}}\mathbb{R}_{(2k\pi,\ (2k+1)\pi)}\right).$

Example 1.13

(a) Find $(0,2)\setminus(1,3)$ and $(1,3)\setminus(0,2)$

(b) Is it generally true that $A\setminus B = B\setminus A$

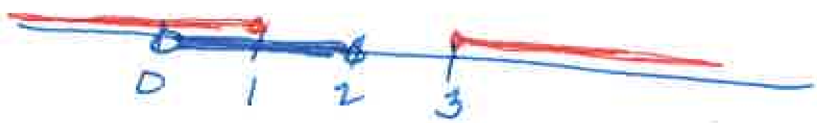
Solution

(a) $(0,2)\setminus(1,3) = \mathbb{R}_{(0,2)} \setminus \mathbb{R}_{(1,3)}$

$= \{x \mid x\in \mathbb{R}_{(0,2)} \text{ and } x\notin \mathbb{R}_{(1,3)}\}$

$= \{x \mid 0<x<2 \text{ and } (x\le 1 \text{ or } x\ge 3)\}$

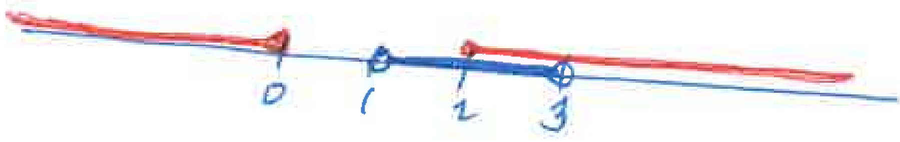
$= \{x \mid 0<x\le 1\} = \mathbb{R}_{(0,1]}$



and

$(1,3)\setminus(0,2) = \mathbb{R}_{(1,3)} \setminus \mathbb{R}_{(0,2)} = \mathbb{R}_{(1,3)} \cap (\mathbb{R}_{(-\infty,0]} \cup \mathbb{R}_{[2,\infty)})$

$= \mathbb{R}_{[2,3)}$



(b) Already (a) shows $(0,2)\setminus(1,3) = \mathbb{R}_{(0,1]}$

and $(1,3)\setminus(0,2) = \mathbb{R}_{[2,3)}$,

so $A\setminus B$ is not generally the same as $B\setminus A$.

Example 1.14 (a) Write the set of real numbers for which $\frac{x^5-2x+4}{x^2-1}$ is defined as both a set complement and as a union of intervals.

Solution $\{x\in\mathbb{R} \mid \frac{x^5-2x+4}{x^2-1} \text{ is defined}\}$

$= \{x\in\mathbb{R} \mid x^2-1\neq 0\} = \{x\in\mathbb{R} \mid x \text{ is not } 1 \text{ or } -1\}$

$= \{1,-1\}^c = \mathbb{R}_{(-\infty,-1)} \cup \mathbb{R}_{(-1,1)} \cup \mathbb{R}_{(1,\infty)}$.

(b) Write the set of real numbers for which $\frac{\log x}{x^2-1}$ is defined as both a set complement and a union of intervals.

Solution

$\{x\in\mathbb{R} \mid \frac{\log x}{x^2-1} \text{ is defined}\}$

$= \{x\in\mathbb{R} \mid \cancel{x\notin\{1,-1\}}\ x^2-1\neq 0 \text{ and } \log x \text{ is defined}\}$

$= \{x\in\mathbb{R} \mid x\notin\{1,-1\} \text{ and } x\in\mathbb{R}_{>0}\}$

$= \{x\in\mathbb{R}_{>0} \mid x\neq 1\} = \{1\}^c$

$= \mathbb{R}_{(0,1)} \cup \mathbb{R}_{(1,\infty)}$.

Example 1.15 Express each of the following as a set complement.

(a) $\{x \in \mathbb{R} \mid x^2 > 1\}$

(b) $\{x \in [-2,2] \mid x \notin \mathbb{Z}\}$

(c) $(-\infty, 0) \cup (0, \infty)$

Solution (a) $\{x \in \mathbb{R} \mid x^2 \geq 1\} = \{x \in \mathbb{R} \mid x \neq 0\} = \{0\}^c$
$= \mathbb{R} \setminus \{0\}$

(b) $\{x \in [-2,2] \mid x \notin \mathbb{Z}\} = \{x \in \mathbb{R}_{[-2,2]} \mid x \notin \{-2,-1,0,1,2\}\}$

$= \cancel{\mathbb{R}_{[-2,1]}} = \cancel{\{x} \; \mathbb{R}_{[-2,2]} \setminus \{-2,-1,0,1,2\}.$

(c) $(-\infty, 0) \cup (0, \infty) = \{x \in \mathbb{R} \mid x < 0 \text{ or } \cancel{\text{and}} \; x > 0\}$

$= \{x \in \mathbb{R} \mid x \neq 0\} = \{0\}^c = \mathbb{R} \setminus \{0\}.$

Example 1.16 List the elements of $A \times B$ where
$A = \{0, 1, 5\}$ and $B = \{e, \pi\}$

Solution

$$A \times B = \left\{ \begin{matrix} (0,e), & (0,\pi) \\ (1,e), & (1,\pi) \\ (5,e), & (5,\pi) \end{matrix} \right\}$$

$$= \{(0,e), (0,\pi), (1,e), (1,\pi), (5,e), (5,\pi)\}.$$