Example 4.51 Let $\frac{dM}{dt} = 0.2M$ where $M$ is the number of $M$ at time $t$.

(a) If there are initially 50 mice find $M(t)$.

(b) When is the mouse population 500?

Solution $M(0) = 50$ and $\frac{dM}{dt} = 0.2M$.

So $\frac{1}{M}\frac{dM}{dt} = 0.2$. So $\int \frac{1}{M}\frac{dM}{dt}\,dt = \int 0.2\,dt$.

So $\log(M) = 0.2t + c_1$, where $c_1$ is a constant.

So $e^{\log(M)} = e^{0.2t+c_1} = e^{0.2t}e^{c_1} = c_2 e^{0.2t}$ where $c_2$ is a constant.

Since $50 = M(0) = c_2 e^{0.2\cdot 0} = c_2$ then $c_2 = 50$.

So $M = 50e^{0.2t}$.

(b) ~~that~~ $500 = M = 50e^{0.2t}$.

Then $10 = e^{0.2t}$. So $\log(10) = 0.2t = \frac{1}{5}t$.

So $5\log(10) = t$.

Example 4.5L Solve $\frac{dP}{dt} = P(1-\frac{P}{4})$ where $P=1$ when $t=0$.

Solution. $\frac{dP}{dt} = \frac{1}{4}P(4-P)$. So $\frac{1}{P(4-P)}\frac{dP}{dt} = \frac{1}{4}$.

So $\left(\frac{\frac{1}{4}}{P} + \frac{\frac{-1}{4}}{4-P}\right)\frac{dP}{dt} = \frac{1}{4}$

So $\int\left(\frac{\frac{1}{4}}{P} + \frac{\frac{-1}{4}}{4-P}\right)\frac{dP}{dt}\,dt = \int\frac{1}{4}\,dt$.

So $\frac{1}{4}\log(P) - \frac{1}{4}\log(4-P) = \frac{1}{4}t + c_1$, where $c_1$ is a constant.

So $\frac{1}{4}\log\left(\frac{P}{4-P}\right) = \frac{1}{4}t + c_1$.

So $\log\left(\left(\frac{P}{4-P}\right)^{\frac{1}{4}}\right) = \frac{1}{4}t + c_1$

So $\left(\frac{P}{4-P}\right)^{\frac{1}{4}} = e^{\frac{1}{4}t + c_1} = e^{\frac{1}{4}t}e^{c_1} = c_2e^{\frac{1}{4}t}$, where $c_2$ is a constant.

So $\frac{P}{4-P} = c_2^4e^t$, where $c_3$ is a
$= c_3e^t$ constant.

So $P = (4-P)c_3e^t = 4c_3e^t - c_3Pe^t$.

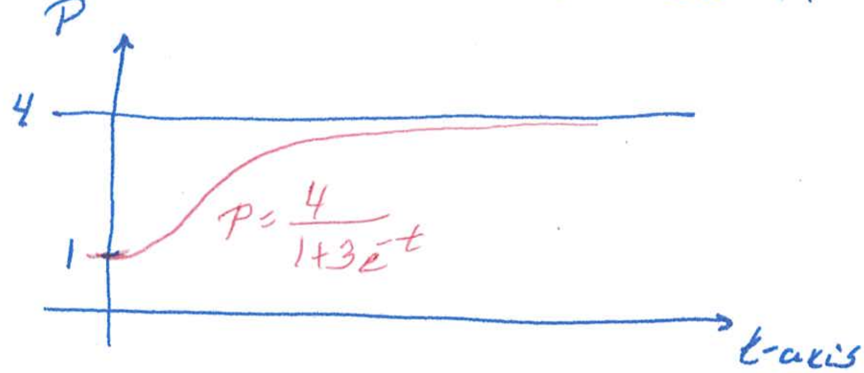
So $P + Pc_3e^t = 4c_3e^t$.

So $P = \frac{4c_3e^t}{1+c_3e^t}$.

Then $1 = P(0) = \dfrac{4C_3e^0}{1+C_3e^0} = \dfrac{4C_3}{1+C_3}$.

So $1 + C_3 = 4C_3$. So $1 = 3C_3$. So $C_3 = \frac{1}{3}$.

So $P = \dfrac{\frac{4}{3}e^t}{1+\frac{1}{3}e^t} = \dfrac{4e^t}{3+e^t} = \dfrac{4}{3e^{-t}+1}$




Example 4.53 The freezer temperature is $-15°C$ and the initial temperature of the bread is $20°C$. It takes 20 min. for the temperature of the bread to drop to $10°C$. How long will it take for the bread to reach $0°C$.

Solution Newton's law of cooling is

$$\frac{dT}{dt} = -k(T - T_s)$$

Here $T_s = -15°$ and $T(0) = 20$ and $T(20) = 10$.

Since

$$\frac{dT}{dt} = -k(T+15) \quad \text{then} \quad \left(\frac{1}{T+15}\right)\frac{dT}{dt} = -k$$

So $\int \left(\frac{1}{T+15}\right)\frac{dT}{dt}\,dt = \int -k\,dt.$

So $\log(T+15) = -kt + c_1$, where $c_1$ is a constant.

So $T+15 = e^{-kt+c_1} = e^{-kt}e^{c_1} = C_2 e^{-kt}$, where $C_2$ is a constant.

So $T = -15 + C_2 e^{-kt}$.

Then $20 = T(0) = -15 + C_2 e^{-k\cdot 0} = -15 + C_2$.

So $C_2 = 35$ and $\cancel{T = 15 + 5e^{-kt}}$ $T = -15 + 35e^{-kt}$

Then $10 = T(20) = -15 + 35e^{-k\cdot 20}$.

So $\frac{25}{35} = e^{-k\cdot 20}$. So $\log\left(\frac{5}{7}\right) = -k\cdot 20$.

So $\frac{1}{20}\log\left(\frac{5}{7}\right) = -k$ and $k = \frac{-1}{20}\log\left(\frac{5}{7}\right)$.

So $T = -15 + 35e^{\frac{1}{20}\log\left(\frac{5}{7}\right)t}$

Then $T=0$ when $0 = -15 + 35e^{\frac{1}{20}\log\left(\frac{5}{7}\right)t}$

So $\frac{15}{35} = e^{\frac{1}{20}\log\left(\frac{5}{7}\right)t}$. So $\log\left(\frac{3}{7}\right) = \frac{1}{20}\log\left(\frac{5}{7}\right)t$.

So $t = \frac{20\log(3/7)}{\log(5/7)}$. when $T=0$.