Example 4.54 The initial value problem for tree growth gives

$$h'(t)=a(1-bh(t)) \text{ and } h(0)=0.$$

Find the solutions and the growth rate at $t=0$.

Solution: Since $h'(0)=a(1-bh(0))=a(1-b\cdot 0)=a$ then the growth rate at $t=0$ is a.

Since $\frac{dh}{dt}=a(1-bh)$ then $\frac{1}{1-bh}\frac{dh}{dt}=a$

and $\int\left(\frac{1}{1-bh}\right)\frac{dh}{dt}\,dt=\int a\,dt.$

So $\frac{1}{-b}\log(1-bh)=at+c_1$, where c_1 is a constant.

So $\log(1-bh)=-abt-bc_1=-abt+c_2$,
where c_2 is a constant.

So $1-bh=e^{-abt+c_2}=e^{-abt}e^{c_2}=C_3e^{-abt}$

where C_3 is a constant.

So $bh=1-C_3e^{-abt}$ and $h=\frac{1}{b}(1-C_3e^{-abt})$.

Since $0=h(0)=\frac{1}{b}(1-C_3e^{-ab\cdot 0})=\frac{1}{b}(1-C_3)$
then $C_3=1$. So $h=\frac{1}{b}(1-e^{-abt})$.

<u>Example 4.55</u> Ents grow to an average maximum height of 20m and at 1 year are average height 2m.

(a) Find the average growth rate of an Ent at birth

(b) Find the average height of an Ent at age 2.

<u>Solution</u> Using the tree growth model

$$\frac{dh}{dt} = a(1-bh) \text{ and } h(0)=0 \text{ then}$$

$$h = \frac{1}{b}(1-e^{-abt}).$$

Since the maximum is 20m then $\frac{1}{b} = 20$ and $b = \frac{1}{20}$.

Since $2 = h(1) = \frac{1}{b}(1-e^{-ab\cdot 1}) = \frac{20}{\cancel{b}}(1-e^{-a\frac{1}{20}})$

~~then~~ $\cancel{40 = 1-e^{-a/20} \text{ and } e^{-a/20} = -39}$

then $\frac{1}{10} = 1-e^{-a/20}$ and $e^{-a/20} = 9/10$.

So $-a/20 = \log(9/10)$ and $a = -20\log(9/10)$

(a) The growth rate a $t=0$ is $a = -20\log(9/10)$

(b) The average height at age 2 is

$$h(2) = 20(1-e^{-a\cdot\frac{1}{20}\cdot 2}) = 20(1-e^{2\log(9/10)})$$

$$= 20\left(1 - e^{\log\left(\left(9/10\right)^2\right)}\right) = 20\left(1 - \left(\frac{9}{10}\right)^2\right)$$

$$= 20\left(\frac{100 - 81}{100}\right) = \frac{1}{5}(19) = \frac{19}{5},$$ slightly under 4m.