Topic 4

Integration

Calculus I
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(1a) $\int (7x^3+6x^2-4x+3)\,dx$

$$\int (7x^3+6x^2-4x+3)\,dx = \frac{7x^4}{4}+\frac{6x^3}{3}-\frac{4x^2}{2}+3x+c$$
$$= \frac{7}{4}x^4+2x^3-2x^2+3x+c, \text{ where } c \text{ is a constant.}$$

(1b) $\int \sin(\pi x)\,dx$

$\int \sin(\pi x)\,dx = \frac{-1}{\pi}\cos(\pi x)+c$, where c is a constant,

since $\frac{d}{dx}\left(\frac{-1}{\pi}\cos(\pi x)+c\right) = \frac{-1}{\pi}(-\sin(\pi x))\pi + 0 = \sin(\pi x)$.

(1c) $\int e^{3x}\,dx$

$\int e^{3x}\,dx = \frac{1}{3}e^{3x}+c$, where c is a constant,

since $\frac{d}{dx}\left(\frac{1}{3}e^{3x}+c\right) = \frac{1}{3}e^{3x}\cdot 3 + 0 = e^{3x}$.

(1d) $\int \frac{3}{2x^2}\,dx$

$\int \frac{3}{2x^2}\,dx = \int \frac{3}{2}x^{-2}\,dx = \frac{3}{2}\cdot\frac{x^{-1}}{-1}+c = \frac{-3}{2x}+c$, where c is a constant.

(1e) $\int \cos\left(\frac{2\pi x}{3}\right)dx$

$\int \cos\left(\frac{2\pi x}{3}\right)dx = \sin\left(\frac{2\pi x}{3}\right)\cdot\frac{3}{2\pi}+c$, where c is a constant,

since $\frac{d}{dx}\left(\frac{3}{2\pi}\sin\left(\frac{2\pi x}{3}\right)+c\right) = \frac{3}{2\pi}\cos\left(\frac{2\pi x}{3}\right)\cdot\frac{2\pi}{3}+0 = \cos\left(\frac{2\pi x}{3}\right)$.

(1f) $\int \frac{4}{x}\,dx$

$\int \frac{4}{x}\,dx = 4\ln x + c$, where c is a constant,

since $\frac{d}{dx}(4\ln x + c) = 4\frac{1}{x}+0 = \frac{4}{x}$.

(1g) $\int \sec^2(\frac{x}{2})dx$

$\int \sec^2(\frac{x}{2})dx = 2\tan(\frac{x}{2}) + c$, where c is a constant,

since $\frac{d}{dx}(2\tan(\frac{x}{2})+c) = 2\sec^2(\frac{x}{2})\cdot\frac{1}{2}+0 = \sec^2(\frac{x}{2})$.

(1h) $\int(2x - \frac{1}{2x})dx$

$\int(2x-\frac{1}{2x})dx = 2\frac{x^2}{2} - 2\ln x + c = x^2 - 2\ln x + c$, where c is a constant,

since $\frac{d}{dx}(x^2 - 2\ln x + c) = 2x - 2\frac{1}{x} + 0 = 2x - \frac{2}{x}$.

(1i) $\int e^{-x/5}dx$

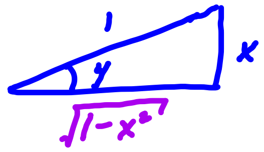
$\int e^{-x/5}dx = -5e^{-x/5} + c$, where c is a constant,

since $\frac{d}{dx}(-5e^{-x/5}+c) = -5e^{-x/5}\cdot(\frac{-1}{5}) + 0 = e^{-x/5}$.

Derivatives of inverse trig functions

Let $y = \arcsin x$. Then $\sin y = x$.

So $\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$. So $\cos y \frac{dy}{dx} = 1$. So $\frac{dy}{dx} = \frac{1}{\cos y}$

If $\sin y = \frac{x}{1}$ then 

and $\cos y = \frac{\sqrt{1-x^2}}{1}$

So $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

So $\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}$

(3a) $\int \frac{-2}{\sqrt{16-x^2}}\,dx$

$$\int \frac{-2}{\sqrt{16-x^2}}\,dx = \int \frac{-2}{\sqrt{16\left(1-\left(\frac{x}{4}\right)^2\right)}}\,dx = \int \frac{-2}{4}\,\frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\,dx$$

$= -2\arcsin\left(\frac{x}{4}\right)+c$, where c is a constant,

since $\frac{d}{dx}\left(-2\arcsin\left(\frac{x}{4}\right)+c\right) = -2\,\frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\cdot\frac{1}{4}+0 = \frac{-2}{4}\,\frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$.

(2b) $\int \frac{4}{25+x^2}\,dx$

$\int \frac{4}{25+x^2}\,dx = \int \frac{4}{25}\cdot\frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx = \frac{4}{25}\arctan\left(\frac{x}{5}\right)\cdot 5 + c = \frac{4}{5}\arctan\left(\frac{x}{5}\right)+c$,
where c is a constant,
because $\frac{d}{dx}\left(\frac{4}{5}\arctan\left(\frac{x}{5}\right)+c\right) = \frac{4}{5}\,\frac{1}{1+\left(\frac{x}{5}\right)^2}\cdot\frac{1}{5}+0 = \frac{4}{25}\cdot\frac{1}{1+\left(\frac{x}{5}\right)^2}$.

(2c) $\int \frac{7}{\sqrt{3-x^2}}\,dx$

$\int \frac{7}{\sqrt{3-x^2}}\,dx = \int \frac{7}{\sqrt{3}}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx = 7\arcsin\left(\frac{x}{\sqrt{3}}\right)+c$, where c is a constant, because

$\frac{d}{dx}\left(7\arcsin\left(\frac{x}{\sqrt{3}}\right)+c\right) = 7\cdot\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\cdot\frac{1}{\sqrt{3}}+0 = \frac{7}{\sqrt{3}}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}$.

(4a) $\int 2x(x^2+1)^5 dx$

$\int 2x(x^2+1)^5 dx = \frac{(x^2+1)^6}{6} + c$, where c is a constant,

since $\frac{d}{dx}\left(\frac{(x^2+1)^6}{6}+c\right) = \frac{1}{6}6(x^2+1)^5 \cdot 2x + 0 = 2x(x^2+1)^5$.

(4b) $\int 3x^2 \cos(x^3+5)\, dx$

$\int 3x^2\cos(x^3+5)dx = \sin(x^3+5)+c$, where c is a constant,

since $\frac{d}{dx}\left(\sin(x^3+5)+c\right) = \cos(x^3+5)\cdot 3x^2 + 0 = 3x^2\cos(x^3+5)$.

(4c) $\int \frac{x}{(x^2+1)^3}\, dx$

$$\int \frac{x}{(x^2+1)^3} dx = \int x(x^2+1)^{-3} dx = \int \tfrac{1}{2}(2x)(x^2+1)^{-3} dx = \frac{1}{2}\frac{(x^2+1)^{-2}}{-2} + c$$

$$= \frac{-1}{4}(x^2+1)^{-2} + c = \frac{-1}{4(x^2+1)^2} + c,$$ where c is a constant,

because $\frac{d}{dx}\left(\frac{-1}{4(x^2+1)^2}+c\right) = \frac{d}{dx}\left(\frac{-1}{4}(x^2+1)^{-2}\right) + 0 = \frac{-1}{4}(-2)(x^2+1)^{-3}2x$

$$= x(x^2+1)^{-3} = \frac{x}{(x^2+1)^3}.$$

(4d) $\int (8x+6)e^{2x^2+3x} dx$

$\int (8x+6)e^{2x^2+3x} dx = \int 2(4x+3)e^{2x^2+3x} dx = 2e^{2x^2+3x} + c$,

where c is a constant.

since $\frac{d}{dx}\left(2e^{2x^2+3x}+c\right) = 2e^{2x^2+3x}(4x+3)+0 = (8x+6)e^{2x^2+3x}$.

(4f) $\int 5x\sqrt{9+x^2}\,dx$

$$\int 5x\sqrt{9+x^2}\,dx = \int \frac{5}{2}(2x)(9+x^2)^{\frac{1}{2}}dx = \frac{5}{2}(9+x^2)^{3/2}\frac{2}{3}+c$$

$$= \frac{5}{3}(9+x^2)^{3/2}+c, \text{ where } c \text{ is a constant,}$$

since $\frac{d}{dx}\left(\frac{5}{3}(9+x^2)^{3/2}+c\right) = \frac{5}{3}\cdot\frac{3}{2}(9+x^2)^{\frac{1}{2}}\cdot 2x + 0$

$$= 5x(9+x^2)^{\frac{1}{2}} = 5x\sqrt{9+x^2}.$$

(4g) $\int \frac{6\sin 3x}{\cos^2 3x}\,dx$

$$\int \frac{6\sin 3x}{\cos^2(3x)}dx = \int 6(\sin 3x)(\cos 3x)^{-2}dx = -6\,\frac{(\cos 3x)^{-1}}{-1}\cdot\frac{1}{3}+c$$

$$= 2\,\frac{1}{\cos 3x} + c, \text{ where } c \text{ is a constant,}$$

since $\frac{d}{dx}\left(\frac{2}{\cos 3x}+c\right) = 2\frac{d}{dx}(\cos 3x)^{-1}+0$

$$= 2(-1)(\cos 3x)^{-2}(-\sin 3x)3 = \frac{6\sin 3x}{\cos^2 3x}.$$

(4h) $\int \frac{5}{x}\sin(\log_e x)\,dx$

$$\int \frac{5}{x}\sin(\log_e x)\,dx = \int -5\,\frac{1}{x}\,-\sin(\ln x)\,dx = -5\cos(\ln x)+c,$$

where c is a constant, since

$$\frac{d}{dx}\left(-5\cos(\ln x)+c\right) = -5\left(-\sin(\ln x)\right)\frac{1}{x}+0 = \frac{5}{x}\sin(\log_e(x)).$$

(5a) $\int \frac{1}{x^2+2x+2}\,dx$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{x^2+2x+1+1}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$$

$= \arctan(x+1)+c$, where c is a constant,

since $\frac{d}{dx}\left(\arctan(x+1)+c\right) = \frac{1}{1+(x+1)^2}+0 = \frac{1}{x^2+2x+2}$.

(5b) $\int \frac{1}{\sqrt{-x^2-4x+2}}\,dx$

$$\int \frac{1}{\sqrt{-x^2-4x+2}}\,dx = \int \frac{1}{\sqrt{-(x^2+4x+4)+6}}\,dx = \int \frac{1}{\sqrt{6-(x+2)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{6}} \frac{1}{\sqrt{1-\left(\frac{(x+2)}{\sqrt{6}}\right)^2}}\,dx = \frac{1}{\sqrt{6}} \arcsin\left(\frac{x+2}{\sqrt{6}}\right)\sqrt{6}+c$$

$= \arcsin\left(\frac{x+2}{\sqrt{6}}\right)+c$, where c is a constant.

since $\frac{d}{dx}\left(\arcsin\left(\frac{x+2}{\sqrt{6}}\right)+c\right) = \frac{1}{\sqrt{1-\left(\frac{x+2}{\sqrt{6}}\right)^2}} \cdot \frac{1}{\sqrt{6}}+0 = \frac{1}{\sqrt{6-(x+2)^2}}$

$$= \frac{1}{\sqrt{6-x^2-4x-4}} = \frac{1}{\sqrt{-x^2-4x+2}}.$$

15c) $\int \frac{-1}{\sqrt{7-x^2+6x}}\,dx$

$$\int \frac{-1}{\sqrt{7-x^2+6x}}\,dx = \int \frac{-1}{\sqrt{16-(x^2-6x+9)}}\,dx = \int \frac{-1}{\sqrt{16-(x-3)^2}}\,dx$$

$$= \int \frac{-1}{4}\,\frac{1}{\sqrt{1-\left(\frac{x-3}{4}\right)^2}}\,dx = -\frac{1}{4}\arcsin\left(\frac{x-3}{4}\right)4 + c,$$

$$= -\arcsin\left(\frac{x-3}{4}\right) + c, \text{ where } c \text{ is a constant,}$$

since $\frac{d}{dx}\left(-\arcsin\left(\frac{x-3}{4}\right)+c\right) = \frac{-1}{\sqrt{1-\left(\frac{x-3}{4}\right)^2}}\,\frac{1}{4} + 0$

$$= \frac{-1}{\sqrt{16-(x-3)^2}} + 0 = \frac{-1}{\sqrt{16-x^2+6x-9}}$$

$$= \frac{-1}{\sqrt{7-x^2+6x}}$$

16a) $\int x\sqrt{1-x}\,dx$

$$\int x\sqrt{1-x}\,dx = \int -(1-x-1)(1-x)^{\frac{1}{2}}dx = \int\left(-(1-x)^{3/2} + (1-x)^{\frac{1}{2}}\right)dx$$
$$= + (1-x)^{5/2}\frac{2}{5} - (1-x)^{3/2}\frac{2}{3} + c = \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + c\right) = \frac{2}{5}\frac{5}{2}(1-x)^{3/2}(-1) - \frac{2}{3}\frac{3}{2}(1-x)^{\frac{1}{2}}(-1) + 0$$
$$= -(1-x)^{3/2} + (1-x)^{\frac{1}{2}} = -(1-x)^{\frac{1}{2}}(1-x-1) = x(1-x)^{\frac{1}{2}} = x\sqrt{1-x}.$$

16b) $\int \frac{2x-1}{(x-1)^2}dx$

$$\int\frac{2x-1}{(x-1)^2}dx = \int\frac{2(x-1)+1}{(x-1)^2}dx = \int\left(\frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$
$$= \int\left(\frac{2}{x-1} + (x-1)^{-2}\right)dx = 2\ln(x-1) + \frac{(x-1)^{-1}}{-1} + c$$
$$= 2\ln(x-1) - \frac{1}{x-1} + c, \text{ where } c \text{ is a constant,}$$

since $\frac{d}{dx}\left(2\ln(x-1) - \frac{1}{x-1} + c\right) = 2\,\frac{1}{x-1} - \frac{d}{dx}\left((x-1)^{-1}\right) + 0$

$$= \frac{2}{x-1} - (-1)(x-1)^{-2} = \frac{2}{x-1} + \frac{1}{(x-1)^2}$$
$$= \frac{2(x-1)}{(x-1)^2} + \frac{1}{(x-1)^2} = \frac{2x-2+1}{(x-1)^2} = \frac{2x-1}{(x-1)^2}.$$

(6c) $\int (x+3)(x+4)^{\frac{1}{2}} dx$

$$\int (x+3)(x+4)^{\frac{1}{2}} dx = \int (x+4-1)(x+4)^{\frac{1}{2}} dx = \int \left((x+4)^{3/2} - (x+4)^{\frac{1}{2}}\right) dx$$

$$= (x+4)^{5/2}\frac{2}{5} - (x+4)^{3/2}\frac{2}{3} + c,$$ where c is a constant,

since $$\frac{d}{dx}\left(\frac{2}{5}(x+4)^{5/2} - \frac{2}{3}(x+4)^{3/2} + c\right) = \frac{2}{5}\cdot\frac{5}{2}(x+4)^{3/2} - \frac{2}{3}\frac{3}{2}(x+4)^{\frac{1}{2}} + 0$$

$$= (x+4)^{3/2} - (x+4)^{\frac{1}{2}} = (x+4)^{\frac{1}{2}}(x+4-1) = (x+4)^{\frac{1}{2}}(x+3).$$

(6d) $\int (2x+1)(x+3)^{20} dx$

$$\int (2x+1)(x+3)^{20} dx = \int (2(x+3)-5)(x+3)^{20} dx$$

$$= \int \left(2(x+3)^{21} - 5(x+3)^{20}\right) dx = \frac{2(x+3)^{22}}{22} - \frac{5(x+3)^{21}}{21} + c$$

$$= \frac{1}{11}(x+3)^{22} - \frac{5}{21}(x+3)^{21} + c,$$ where c is a constant,

since $$\frac{d}{dx}\left(\frac{1}{11}(x+3)^{22} - \frac{5}{21}(x+3)^{21} + c\right) = \frac{1}{11}\cdot 22(x+3)^{21} - \frac{5}{21}\cdot 21(x+3)^{20}$$

$$= 2(x+3)^{21} - 5(x+3)^{20} = (x+3)^{20}(2(x+3)-5)$$

$$= (x+3)^{20}(2x+1).$$

(8a) $\displaystyle\int \frac{3x}{(x-2)(x+4)}\,dx$

$\dfrac{3x}{(x-2)(x+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+4}$ so that $\begin{array}{l} A+B=3 \\ 4A-2B=0 \end{array}$ giving

$\begin{array}{l} 2A=B \\ A+2A=3 \end{array}$ so that $\begin{array}{l} A=1 \\ B=2 \end{array}$ and

$\displaystyle\int \frac{3x}{(x-2)(x+4)}\,dx = \int\left(\frac{1}{x-2} + \frac{2}{x+4}\right)dx = \ln(x-2) + 2\ln(x+4) + c,$

where c is a constant.

(8b) $\displaystyle\int \frac{3}{x^2-5x+4}\,dx$

$\dfrac{3}{x^2-5x+4} = \dfrac{3}{(x-4)(x-1)} = \dfrac{A}{x-4} + \dfrac{B}{x-1}$ with $\begin{array}{l} A+B=0 \\ -A-4B=3 \end{array}$

so that $\begin{array}{l} B=-A \\ B-4B=3, \end{array}$ giving $\begin{array}{l} -3B=3 \\ B=-1,\ A=1. \end{array}$

So $\displaystyle\int \frac{3}{x^2-5x+4}\,dx = \int\left(\frac{1}{x-4} - \frac{1}{x-1}\right)dx = \ln(x-4) - \ln(x-1) + c,$

where c is a constant.

(8c) $\int \frac{2x+1}{x^2+4x+4}\,dx$

$$\int \frac{2x+1}{x^2+4x+4}\,dx = \int \frac{2x+1}{(x+2)^2}\,dx = \int \frac{2(x+2)-3}{(x+2)^2}\,dx$$

$$= \int \left(\frac{2}{x+2} - \frac{3}{(x+2)^2}\right) dx = \int \left(\frac{2}{x+2} - 3(x+2)^{-2}\right) dx$$

$$= 2\ln(x+2) - \frac{3(x+2)^{-1}}{(-1)} + c$$

$$= 2\ln(x+2) + \frac{3}{x+2} + c,$$ where c is a constant,

since $\frac{d}{dx}\left(2\ln(x+2) + 3(x+2)^{-1} + c\right)$

$$= \frac{2}{x+2} + 3(-1)(x+2)^{-2} + 0$$

$$= \frac{2}{x+2} - \frac{3}{(x+2)^2} = \frac{2(x+2)}{(x+2)^2} - \frac{3}{(x+2)^2}$$

$$= \frac{2x+4-3}{(x+2)^2} = \frac{2x+1}{(x+2)^2}.$$

(8d) $\int \frac{x+3}{x^2-3x+2}\,dx$

$$\int \frac{x+3}{x^2-3x+2}\,dx = \int \frac{x+3}{(x-2)(x-1)}\,dx = \int\left(\frac{A}{x-2}+\frac{B}{x-1}\right)dx$$

with $\begin{matrix} A+B=1 \\ -2B-A=3 \end{matrix}$ so that $\begin{matrix} B=1-A \\ -2(1-A)-A=3 \end{matrix}$

So $\begin{matrix} B=1-A \\ A=5 \end{matrix}$ and $B=-4$.

$$\int\left(\frac{5}{x-2}-\frac{4}{x-1}\right)dx = 5\ln(x-2)-4\ln(x-1)+c,$$

where c is a constant,

since $\frac{d}{dx}\left(5\ln(x-2)-4\ln(x-1)+c\right)$

$$= \frac{5}{x-2}-\frac{4}{x-1}+0 = \frac{5(x-1)-4(x-2)}{(x-2)(x-1)}$$

$$= \frac{5x-5-4x+8}{x^2-3x+2} = \frac{x+3}{x^2-3x+2}.$$

(8e) $\int \frac{2x+1}{x^2-1}\,dx$

$$\int \frac{2x+1}{x^2-1}\,dx = \int \frac{2x+1}{(x-1)(x+1)}\,dx = \int\left(\frac{A}{x-1}+\frac{B}{x+1}\right)dx$$

with $\begin{matrix} A+B=2 \\ A-B=1 \end{matrix}$ so that $\begin{matrix} B=2-A \\ A-(2-A)=1 \end{matrix}$ and

$\begin{matrix} B=2-A \\ 2A=3 \end{matrix}$ and $\begin{matrix} A=\frac{3}{2} \\ B=\frac{1}{2} \end{matrix}$

$$\int\left(\frac{3/2}{x-1}+\frac{\frac{1}{2}}{x+1}\right)dx = \frac{3}{2}\ln(x-1)+\frac{1}{2}\ln(x+1)+c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{3}{2}\ln(x-1)+\frac{1}{2}\ln(x+1)+c\right)$$

$$= \frac{3/2}{x-1}+\frac{1/2}{x+1}+0 = \frac{\frac{3}{2}(x+1)+\frac{1}{2}(x-1)}{(x-1)(x+1)}$$

$$= \frac{\left(\frac{3}{2}+\frac{1}{2}\right)x+\left(\frac{3}{2}-\frac{1}{2}\right)}{x^2-1} = \frac{2x+1}{x^2-1}.$$

(8f) $\displaystyle\int \frac{2x}{x^2-2x+10}\,dx$

$$\int \frac{2x}{x^2-2x+10}\,dx = \int \frac{2x-2+2}{x^2-2x+10}\,dx$$

$$= \int\left(\frac{2x-2}{x^2-2x+10} + \frac{2}{x^2-2x+1+9}\right)dx$$

$$= \int\left(\frac{2x-2}{x^2-2x+10} + \frac{2}{(x-1)^2+9}\right)dx$$

$$= \ln(x^2-2x+10) + \frac{2}{3}\arctan\left(\frac{x-1}{3}\right) + c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\ln(x^2-2x+10) + \frac{2}{3}\arctan\left(\frac{x-1}{3}\right) + c\right)$$

$$= \frac{1}{x^2-2x+10}(2x-2) + \frac{2}{3}\cdot\frac{1}{1+\left(\frac{x-1}{3}\right)^2}\cdot\frac{1}{3} + 0$$

$$= \frac{2x-2}{x^2-2x+10} + \frac{2}{3^2\left(\frac{3^2+(x-1)^2}{3^2}\right)} = \frac{2x-2+2}{x^2-2x+10}$$

$$= \frac{2x}{x^2-2x+10}.$$

19a) $\displaystyle\int \frac{x^3+x^2-3x+3}{x+2}\,dx$

$$\int \frac{x^3+x^2-3x+3}{x+2}\,dx = \int \frac{(x+2)(x^2-x-1)+5}{x+2}\,dx$$

$$= \int\left((x^2-x-1)+\frac{5}{x+2}\right)dx = \frac{x^3}{3}-\frac{x^2}{2}-x$$

$$+5\ln(x+2)+c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{x^3}{3}-\frac{x^2}{2}-x+5\ln(x+2)+c\right)$$

$$= x^2-x-1+\frac{5}{x+2} = \frac{(x+2)(x^2-x-1)+5}{x+2}$$

$$= \frac{x^3-x^2-x+2x^2-2x-2+5}{x+2} = \frac{x^3+x^2-3x+3}{x+2}.$$

(96) $\displaystyle\int \frac{4x^3+8x^2+5x+13}{4x^2+5}\,dx$

$$\int \frac{4x^3+8x^2+5x+13}{4x^2+5}\,dx$$

$$= \int \frac{(4x^2+5)(x+2)+3}{4x^2+5}\,dx = \int \left(x+2+\frac{3}{4x^2+5}\right)dx$$

$$= \int \left(x+2+\frac{3}{5}\left(\frac{1}{\left(\frac{2x}{\sqrt{5}}\right)^2+1}\right)\right)dx$$

$$= \frac{x^2}{2}+5x+\frac{3}{5}\arctan\left(\frac{2x}{\sqrt{5}}\right)\frac{\sqrt{5}}{2}+c$$

$$= \frac{x^2}{2}+5x+\frac{3\sqrt{5}}{10}\arctan\left(\frac{2x}{\sqrt{5}}\right)+c,$$

where c is a constant.

(9c) $\int \frac{x^3+3}{x^2-x}\,dx$

$$\int \frac{x^3+3}{x^2-x}\,dx = \int \frac{(x^2-x)(x+1)+x+3}{x^2-x}\,dx$$

$$= \int\left(x+1+\frac{x+3}{x(x-1)}\right)dx = \int\left(x+1+\frac{A}{x}+\frac{B}{x-1}\right)dx$$

with $\begin{matrix} A+B=1 \\ -A=3 \end{matrix}$ so that $\begin{matrix} B=4 \\ A=-3 \end{matrix}$

$$\int\left(x+1+\frac{-3}{x}+\frac{4}{x-1}\right)dx = \frac{x^2}{2}+x-3\ln x$$
$$+4\ln(x-1)+c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{x^2}{2}+x-3\ln x+4\ln(x-1)+c\right)$$

$$= x+1-\frac{3}{x}+\frac{4}{x-1}+0$$

$$= \frac{(x+1)x(x-1)-3(x-1)+4x}{x(x-1)} = \frac{x^3-x-3x+3+4x}{x^2-x}$$

$$= \frac{x^3+3}{x^2-x}$$

(9d) $\displaystyle\int \frac{x^3+4x^2+11x+2}{x^2+4x+10}\,dx$

$$\int \frac{x^3+4x^2+11x+2}{x^2+4x+10}\,dx = \int \frac{(x^2+4x+10)x + x+2}{x^2+4x+10}\,dx$$

$$= \int \left(x + \frac{x+2}{x^2+4x+10}\right) dx = \int \left(x + \frac{\frac{1}{2}(2x+4)}{x^2+4x+10}\right) dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\ln(x^2+4x+10) + c,$$ where c is a constant.

(9e) $\displaystyle\int \frac{4x^3+42x+1}{x^2+10}\,dx$

$$\int \frac{4x^3+42x+1}{x^2+10}\,dx = \int \frac{(x^2+10)\,4x + 2x+1}{x^2+10}\,dx$$

$$= \int \left(4x + \frac{2x}{x^2+10} + \frac{1}{10}\left(\frac{1}{\left(\frac{x}{\sqrt{10}}\right)^2+1}\right)\right) dx$$

$$= 2x^2 + \ln(x^2+10) + \frac{\sqrt{10}}{10}\arctan\left(\frac{x}{\sqrt{10}}\right) + c,$$

where c is a constant.

(7a) $\int 2\sin^4 2x \cos 2x \, dx$

$$\int 2\sin^4 2x \cos 2x \, dx = \frac{1}{5}\sin^5 2x + c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{1}{5}(\sin 2x)^5 + c\right) = \frac{1}{5}\cdot 5(\sin 2x)^4 \cos 2x \cdot 2 + 0$$

$$= 2\sin^4 2x \cos 2x.$$

(7b) $\int \sin^2 x \, dx$

$$\int \sin^2 x \, dx = \int \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 dx = \int \left(\frac{e^{i2x} - 2 + e^{-i2x}}{-4}\right) dx$$

$$= \frac{-1}{4}\left(\frac{e^{i2x}}{2i} - 2x + \frac{e^{-i2x}}{-2i}\right) + c$$

$$= \frac{-1}{4}\left(\frac{e^{i2x} - e^{-i2x}}{2i} - 2x\right) + c = \frac{-1}{4}\sin 2x + \frac{x}{2} + c,$$

where c is a constant.

(7c) $\int \cos^5 2x \, dx$

$$\int \cos^5 2x \, dx = \int (\cos^4 2x)(\cos 2x)\, dx$$

$$= \int \tfrac{1}{2} (\cos^2 2x)^2 (\cos 2x) \cdot 2 \, dx$$

$$= \int \tfrac{1}{2} (1 - \sin^2 2x)^2 \cos(2x) \cdot 2 \, dx$$

$$= \int \tfrac{1}{2} (1 - 2\sin^2 2x + \sin^4 2x) \cos 2x \cdot 2 \, dx$$

$$= \tfrac{1}{2} \left(\sin 2x - \tfrac{2}{3} \sin^3 2x + \sin^5 2x \right) + c$$

$$= \tfrac{1}{2} \sin 2x - \tfrac{1}{3} \sin^3 2x + \sin^5 2x + c,$$

where c is a constant.

(7d) $\int \cos^2(7x)\,dx$

$$\int \cos^2 7x\,dx = \int \left(\frac{e^{i7x}+e^{-i7x}}{2}\right)^2 dx$$

$$= \int \frac{1}{4}\left(e^{i14x}+2+e^{-i14x}\right) dx$$

$$= \frac{1}{4}\left(\frac{e^{i14x}}{14i} + 2x + \frac{e^{-i14x}}{-14i}\right) + c$$

$$= \frac{1}{7\cdot 4}\left(\frac{e^{i14x}-e^{-i14x}}{2i} + 2x\right) + c$$

$$= \frac{1}{28}\sin(14x) + \frac{1}{14}x + c,$$ where c is a constant.

(7e) $\int \cos^2 5x \sin^2 5x\,dx$

$$\int \cos^2 5x \sin^2 5x\,dx = \int \left(\frac{e^{i5x}+e^{-i5x}}{2}\right)^2\left(\frac{e^{i5x}-e^{-i5x}}{2i}\right)^2 dx$$

$$= \int \frac{\left(e^{i10x}-e^{-i10x}\right)^2}{4(-4)}\,dx = \int \frac{e^{i20x}-2+e^{-i20x}}{-16}\,dx$$

$$= -\frac{1}{16}\left(\frac{e^{i20x}}{20i} - 2x + \frac{e^{-i20x}}{-20i}\right) + c$$

$$= \frac{1}{8}x - \frac{1}{16\cdot 10}\left(\frac{e^{i20x} - e^{-i20x}}{2i}\right) + c$$

$$= -\frac{1}{8}x - \frac{1}{160}\sin(20x) + c,$$ where c is a constant.

(7f) $\int \sin^4 x \cos^5 x \, dx$

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \left(1 - \sin^2 x\right)^2 \cos x \, dx$$

$$= \int \sin^4 x \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x \, dx$$

$$= \int \left(\sin^4 x - 2\sin^6 x + \sin^8 x\right) \cos x \, dx$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + c,$$

where c is a constant.

(10a) $\int \cos^2 x \sin^3 x \, dx$

$$\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x \sin x \, dx$$

$$= \int -\cos^2 x (1 - \cos^2 x)(-\sin x) \, dx$$

$$= \int (-\cos^2 x + \cos^4 x)(-\sin x) \, dx$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c,$$ where c is a constant.

(10b) $\int \frac{x^3 + 3x - 2}{x^2 - x} \, dx$

$$\int \frac{x^3 + 3x - 2}{x^2 - x} \, dx = \int \frac{(x^2 - x)(x + 1) + 4x - 2}{x^2 - x} \, dx$$

$$= \int \left(x + 1 + \frac{2(2x - 1)}{x^2 - x} \right) dx = x + 1 + 2 \ln(x^2 - x) + c,$$

where c is a constant.

(10c) $\int \cos 2x \, e^{\sin 2x} dx$

$$\int \cos 2x \, e^{\sin 2x} dx = \frac{1}{2} e^{\sin 2x} + c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{1}{2} e^{\sin 2x} + c\right) = \frac{1}{2} e^{\sin 2x} \cos 2x \cdot 2 + 0$$

$$= \cos 2x \, e^{\sin 2x}.$$

(10d) $\int \cos^4 x \, dx$

$$\int \cos^4 x \, dx = \int \left(\frac{e^{ix} - e^{-ix}}{2}\right)^4 dx$$

$$= \int \frac{1}{2^4}\left(e^{i4x} - 4e^{i2x} + 6 - 4e^{-i2x} + e^{-i4x}\right)$$

$2x$

$$= \frac{1}{2^4}\left(\frac{e^{i4x}}{4i} - \frac{4}{2i} e^{i2x} + 6x - \frac{4}{-2i} e^{-i2x} + \frac{e^{-i4x}}{-4i}\right) + c$$

$$= \frac{1}{2^4}\left(\frac{1}{2}\left(\frac{e^{i4x}-e^{-i4x}}{2i}\right)-4\left(\frac{e^{i2x}-e^{-i2x}}{2i}\right)\right.$$
$$\left. + 6x + c\right)$$

$$= \frac{1}{2^5}\sin 4x - \frac{1}{2^2}\sin 2x + \frac{3x}{2^4} + c,$$

where c is a constant.

(10d) $\int \tan^2 5x \sec^2 5x \, dx$

$$\int \tan^2 5x \sec^2 5x = \frac{1}{15}\tan^3 5x + c,$$

where c is a constant, since

$$\frac{d}{dx}\left(\frac{1}{15}(\tan 5x)^3 + c\right) =$$

$$= \frac{1}{15}\, 3\tan^2 5x \cdot \sec^2 5x \cdot 5 + 0$$

$$= \tan^2 5x \sec^2 5x.$$

(10f) $\displaystyle\int \frac{1}{\sqrt{5+x}}\,dx$

$$\int \frac{1}{\sqrt{5+x}}\,dx = \int (5+x)^{-\frac{1}{2}}\,dx$$

$$= (5+x)^{\frac{1}{2}} \cdot 2 + c,$$ where c is a constant

since $\displaystyle\frac{d}{dx}\left((5+x)^{\frac{1}{2}} \cdot 2 + c\right)$

$$= \frac{1}{2}(5+x)^{-\frac{1}{2}} \cdot 2 + 0$$

$$= (5+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{5+x}}$$

(10g) $\int \frac{-2x-3}{x^2-x}\,dx$

$$\int \frac{-2x-3}{x^2-x}\,dx = \int\left(\frac{A}{x} + \frac{B}{x-1}\right)dx, \text{ with } \begin{array}{l} A+B=-2 \\ -A=-3 \end{array}$$

so that $A=3$, $B=-5$.

$$\int\left(\frac{3}{x} + \frac{-5}{x-1}\right)dx = 3\ln x - 5\ln(x-1) + c,$$ where c is a constant.

(10h) $\int \frac{x-2}{\sqrt{x+1}}\,dx$

$$\int \frac{x-2}{\sqrt{x+1}}\,dx = \int (x-2)(x+1)^{-\frac{1}{2}}\,dx = \int (x+1-3)(x+1)^{-\frac{1}{2}}\,dx$$

$$= \int\left((x+1)^{\frac{1}{2}} - 3(x+1)^{-\frac{1}{2}}\right)dx = \frac{2}{3}(x+1)^{3/2} - 3\cdot 2(x+1)^{\frac{1}{2}} + c,$$

where c is a constant.

(10i) $\int \frac{1}{8+2x^2}\,dx$

$$\int \frac{1}{8+2x^2}\,dx = \int \frac{1}{8\left(1+\frac{x^2}{4}\right)}\,dx = \int \frac{1}{8}\,\frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{8}\cdot 2\arctan\left(\frac{x}{2}\right) + c = \frac{1}{4}\arctan\left(\frac{x}{2}\right) + c,$$ where c is a constant.

(10j) $\int \frac{1}{x^2+6x+10}\,dx$

$$\int \frac{1}{x^2+6x+10}\,dx = \int \frac{1}{x^2+6x+9+1}\,dx = \int \frac{1}{(x+3)^2+1}\,dx$$

$= \arctan(x+3) + c$, where c is a constant.

(10k) $\int \frac{x^4+6x+10x^2+11x+13}{x^2+6x+10}\,dx$

$$\int \frac{x^4+6x^3+10x^2+11x+13}{x^2+6x+10}\,dx = \int \frac{(x^2+6x+10)x^2+11x+13}{x^2+6x+10}\,dx$$

$$= \int \left(x^2 + \frac{\frac{11}{2}(2x+6)-20}{x^2+6x+10}\right) dx = \int \left(x^2 + \frac{1}{2}\cdot\frac{2x+6}{x^2+6x+10} - 20\,\frac{1}{(x+3)^2+1}\right) dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}\ln(x^2+6x+10) - 20\arctan(x+3) + c,$$

where c is a constant.

(11a) $\int_0^2 (3x^2+2x+4)\,dx$

$$\int_0^2 (3x^2+2x+4)\,dx = x^3+x^2+4x\Big]_{x=0}^{x=2} = (8+4+8)-(0+0+0)$$

$= 20$

(11b) $\int_0^{\pi/4} (\cos\theta+2\theta)\,d\theta$

$$\int_0^{\pi/4} (\cos\theta+2\theta)\,d\theta = \sin\theta+\theta^2\Big]_{\theta=0}^{\theta=\pi/4} = \left(\sin\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2\right)-\left(\sin 0+0^2\right)$$

$= \frac{\sqrt{2}}{2}+\frac{\pi^2}{16}-0-0 = \frac{\sqrt{2}}{2}+\frac{\pi^2}{16}.$

(11c) $\int_{-1}^1 (e^t-e^{-t})\,dt$

$$\int_{-1}^1 (e^t-e^{-t})\,dt = e^t+e^{-t}\Big]_{t=-1}^{t=1} = (e+e^{-1})-(e^{-1}+e) = 0.$$

(11d) $\int_1^2 \frac{1}{3t}\,dt$

$$\int_1^2 \frac{1}{3t}\,dt = \frac{1}{3}\ln t\Big]_{t=1}^{t=2} = \frac{1}{3}\ln 2-\frac{1}{3}\ln 1 = \frac{1}{3}\ln 2 - 0 = \frac{1}{3}\ln 2.$$

(12a) $\int_0^1 x\sqrt{1-x}\,dx$

$$\int_0^1 x\sqrt{1-x}\,dx = \int_0^1 x(1-x)^{\frac{1}{2}}dx = \int_0^1 -(-1+1-x)(1-x)^{\frac{1}{2}}dx$$

$$= \int_0^1 \left((1-x)^{\frac{1}{2}} - (1-x)^{3/2}\right)dx = \frac{2}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2}\Big]_{x=0}^{x=1}$$

$$= \left(\frac{2}{3}(1-1)^{3/2} - \frac{2}{5}(1-1)^{5/2}\right) - \left(\frac{2}{3}(1-0)^{3/2} - \frac{2}{5}(1-0)^{5/2}\right)$$

$$= 0 - 0 - \left(\frac{2}{3} - \frac{2}{5}\right) = -\frac{4}{15}.$$

(12b) $\int_0^{\pi/4} \sin^3 t \cos t\,dt$

$$\int_0^{\pi/4} \sin^3 t \cos t\,dt = \frac{1}{4}\sin^4 t\Big]_{t=0}^{t=\pi/4} = \frac{1}{4}\left(\sin\frac{\pi}{4}\right)^4 - \frac{1}{4}(\sin 0)^4$$

$$= \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)^4 - 0 = \frac{2^2}{2^6} = \frac{1}{2^4} = \frac{1}{16}.$$

(12c) $\int_0^3 x\sqrt{x^2+16}\,dx$

$$\int_0^3 x\sqrt{x^2+16}\,dx = \int_0^3 \frac{1}{2}2x(x^2+16)^{\frac{1}{2}}dx = \frac{1}{2}\cdot\frac{2}{3}(x^2+16)^{3/2}\Big]_{x=0}^{x=3}$$

$$= \frac{1}{3}(3^2+16)^{3/2} - \frac{1}{3}(0^2+16)^{3/2} = \frac{1}{3}5^3 - \frac{1}{3}4^3 = \frac{1}{3}(125-64) = \frac{61}{3}.$$

(12d) $\int_e^{e^2} \frac{1}{\log x} \cdot \frac{1}{x}\, dx$

$$\int_e^{e^2} \frac{1}{\log x} \cdot \frac{1}{x}\, dx = \log(\log x)\Big]_{x=e}^{x=e^2} = \log(\log e^2) - \log(\log e)$$

$$= \log 2 - \log 1 = \ln 2.$$

(13a) $\int_2^3 \frac{3}{x^3}\, dx$

$$\int_2^3 \frac{3}{x^3}\, dx = \int_{x=2}^{x=3} 3x^{-3}\, dx = 3\frac{1}{-2}x^{-2}\Big]_{x=2}^{x=3} = \frac{-2}{3}\frac{1}{3^2} - \frac{-2}{3}\frac{1}{2^2}$$

$$= \frac{-2}{3^3} + \frac{1}{2\cdot 3} = \frac{-4+3^2}{2\cdot 3^3} = \frac{5}{54}.$$

(13b) $\int_0^{\pi/2} \sin^2 2\theta\, d\theta$

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{e^{i2\theta} - e^{-i2\theta}}{2i}\right)^2 d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{-1}{4}\left(e^{i4\theta} - 2 + e^{-i4\theta}\right) d\theta$$

$$= \frac{-1}{4}\left(\frac{e^{i4\theta}}{4i} - 2\theta + \frac{e^{-i4\theta}}{-4i}\right)\Big]_{\theta=0}^{\theta=\pi/4} = \frac{-1}{4}\left(\frac{e^{i4\theta} - e^{-i4\theta}}{2i}\right)\frac{1}{2} + \frac{\theta}{2}\Big]_{\theta=0}^{\theta=\frac{\pi}{4}}$$

$$= \frac{-1}{8}\sin 4\theta + \frac{\theta}{2}\Big]_{\theta=0}^{\theta=\frac{\pi}{4}} = \left(\frac{-1}{8}\sin\pi + \frac{\pi}{8}\right) - \left(\frac{-1}{8}\sin 0 + 0\right) = 0 + \frac{\pi}{8} = \frac{\pi}{8}.$$

(13c) $\int_0^{1/2} \frac{3}{\sqrt{1-x^2}}\,dx$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\sqrt{1-\sin^2\theta}} \frac{d(\sin\theta)}{d\theta}\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{1}{\sqrt{\cos^2\theta}} \cos\theta\,d\theta$$

(since $\sin\frac{\pi}{6} = \frac{1}{2}$)

$$= \int_{\theta=0}^{\theta=\pi/6} d\theta = \theta \Big]_{\theta=0}^{\theta=\frac{\pi}{6}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

(13d) $\int_1^2 \frac{1}{x(x+2)}\,dx$

$$\int_1^2 \frac{1}{x(x+2)}\,dx = \int_{x=1}^{x=2} \left(\frac{A}{x} + \frac{B}{x+2}\right)dx \quad \text{with} \quad \begin{array}{l} A+B=0 \\ 2A=1 \end{array}$$

$$= \int_{x=1}^{x=2} \left(\frac{1}{2}\cdot\frac{1}{x} + \frac{-1}{2}\left(\frac{1}{x+2}\right)\right)dx = \frac{1}{2}\ln x - \frac{1}{2}\ln(x+2)\Big]_{x=1}^{x=2}$$

$$= \left(\frac{1}{2}\ln 2 - \frac{1}{2}\ln 3\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{2}\ln 2\right) = \ln 2 - \frac{1}{2}\ln 3.$$

(13e) $\int_2^3 \frac{2x+6}{(x-1)^2}\,dx$

$$\int_2^3 \frac{2x+6}{(x-1)^2}\,dx = \int_2^3 \frac{2(x-1)+8}{(x-1)^2}\,dx = \int_2^3 \left(2(x-1)\frac{1}{(x-1)^2} + 8(x-1)^{-2}\right)dx$$

$$= \ln((x-1)^2) + 8\frac{1}{-1}(x-1)^{-1}\Big]_{x=2}^{x=3}$$

$$= \ln(x-1)^2 - \frac{8}{x-1}\Big]_{x=2}^{x=3} = \left(\ln(2^2) - \frac{8}{2}\right) - \left(\ln(1^2) - \frac{8}{1}\right)$$

$$= \ln 4 - 4 + 8 = 4 + \ln 4.$$

(13f) $\int_{-1}^{1} \frac{e^t}{e^t+1}\,dt$

$$\int_{-1}^{1} \frac{e^t}{e^t+1}\,dt = \ln(e^t+1)\Big]_{t=-1}^{t=1} = \ln(e+1) - \ln(e^{-1}+1)$$

(13g) $\int_{0}^{\pi/4} \sec^2 x\,dx$

$$\int_{0}^{\pi/4} \sec^2 x\,dx = \tan x\Big]_{x=0}^{x=\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1.$$

(13h) $\int_{0}^{1} \frac{x^3+x^2+4x+1}{x^2+1}\,dx$

$$\int_{0}^{1} \frac{x^3+x^2+4x+1}{x^2+1}\,dx = \int_{0}^{1}\left(\frac{(x^2+1)x}{x^2+1} + \frac{2\cdot 2x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{2}x^2 + 2\cdot\ln(x^2+1) + \arctan x\Big]_{x=0}^{x=1}$$

$$= \left(\frac{1}{2} + 2\ln 2 + \arctan 1\right) - (0 + 2\ln 1 + \arctan 0)$$

$$= \frac{1}{2} + 2\ln 2 + \frac{\pi}{4} - 0 = \frac{\pi}{4} + \frac{1}{2} + 2\ln 2.$$

(13 i) $\displaystyle\int_{-\pi/6}^{\pi/3} \frac{\sin x}{10+6\cos x+\cos^2 x}\, dx$

$$\int_{-\pi/6}^{\pi/3} \frac{\sin x}{10+6\cos x+\cos^2 x}\, dx = \int_{x=-\pi/6}^{x=\pi/3} \frac{1}{10+6y+y^2}\, dy = \int_{y=\frac{\sqrt{3}}{2}}^{y=\frac{1}{2}} \frac{1}{(y+3)^2+1}\, dy$$

(since $\cos\left(\frac{-\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$)

$$= \arctan(y+3)\Big|_{y=\frac{\sqrt{3}}{2}}^{y=\frac{1}{2}} = \arctan\left(\frac{7}{2}\right) - \arctan\left(3+\frac{\sqrt{3}}{2}\right).$$

(14a) Find the area under $y = x + \frac{1}{x^2}$ from $x=2$ to $x=3$.

$$\begin{pmatrix}\text{Area under } y = x + \frac{1}{x^2} \\ \text{from } x=2 \text{ to } x=3\end{pmatrix} = \int_{x=2}^{x=3} \left(x + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 + \frac{1}{-1}x^{-1} \Big]_{x=2}^{x=3}$$

$$= \left(\frac{1}{2}3^2 - \frac{1}{3}\right) - \left(\frac{1}{2}\cdot 2^2 - \frac{1}{2}\right) = \frac{6}{2} - \frac{1}{3} = \frac{8}{3}.$$

(14b) Find the area under $y = \frac{6}{4+x^2}$ from $x=2$ to $x=2$.

$$\begin{pmatrix}\text{Area under } y = \frac{6}{4+x^2} \\ \text{from } x=-2 \text{ to } x=2\end{pmatrix} = \int_{x=-2}^{x=2} \frac{6}{4+x^2}\, dx = \int_{x=-2}^{x=2} \frac{6}{4} \frac{1}{1+\left(\frac{x}{2}\right)^2}\, dx$$

$$= \frac{3}{2}\cdot 2 \arctan\left(\frac{x}{2}\right)\Big]_{x=-2}^{x=2} = 3\arctan 1 - 3\arctan(-1)$$

$$= 3\frac{\pi}{4} - 3\left(\frac{-\pi}{4}\right) = \frac{6\pi}{4} = \frac{3\pi}{2}.$$

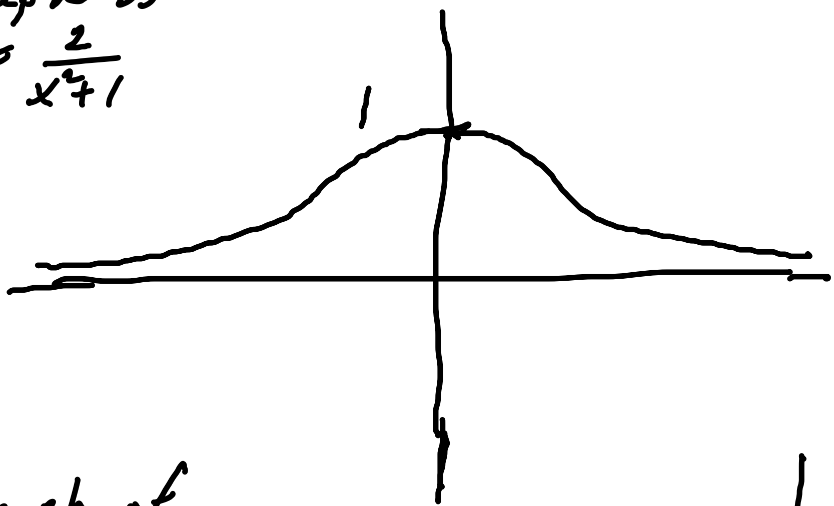
(14c) Find the area under $y = \frac{2}{x}\log x$ from $x=1$ to $x=e$

$$\begin{pmatrix}\text{Area under } y = \frac{2}{x}\log x \\ \text{from } x=1 \text{ to } x=e\end{pmatrix} = \int_{x=1}^{x=e} \frac{2}{x}\log x\, dx = 2\,\frac{(\log x)^2}{2}\Big]_{x=1}^{x=e}$$

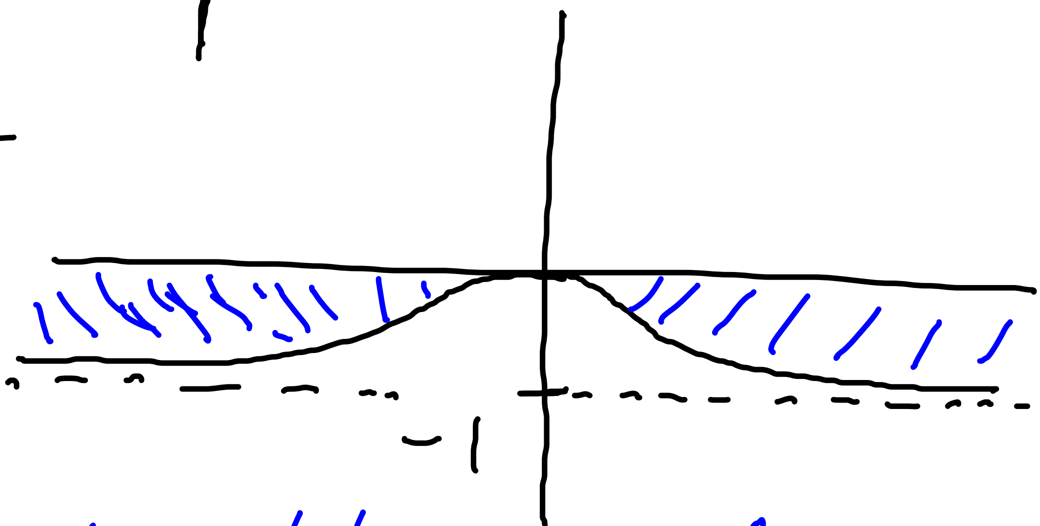
$$= (\log e)^2 - (\log 1)^2 = 1^2 - 0^2 = 1.$$

(14d) Find the area under $y = -1 + \frac{2}{x^2+1}$ and the x-axis.

Graph of
$y = \frac{2}{x^2+1}$



Graph of
$y = -1 + \frac{2}{x^2+1}$



shaded area is area between $y = -1 + \frac{2}{x^2+1}$ and the x-axis.

This is infinite.

We can arrive at the same conclusion by integration:

$$\begin{pmatrix} \text{Area between } y = -1 + \frac{2}{x^2+1} \\ \text{and the } x\text{-axis} \end{pmatrix} = \int_{x=-\infty}^{x=\infty} \left(-1 + \frac{2}{x^2+1}\right) dx = 2\int_{x=0}^{x=\infty} \left(-1 + \frac{2}{x^2+1}\right) dx$$

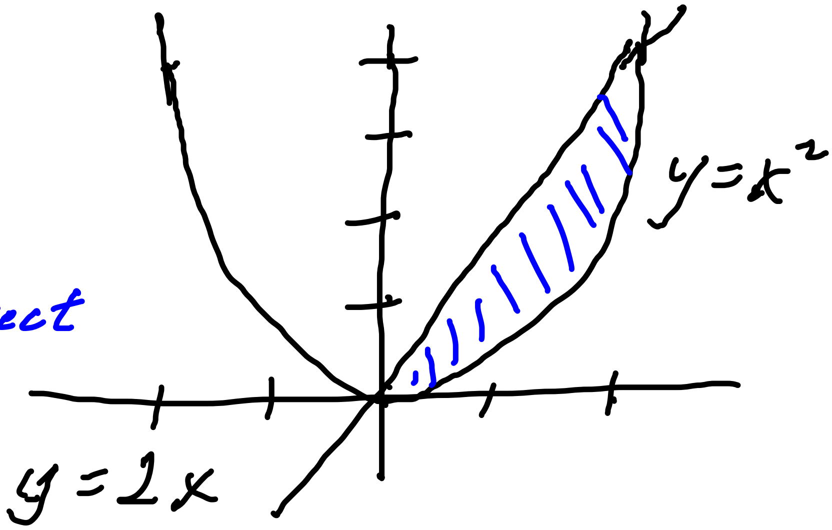
$$= 2\left(-x + 2\arctan x\right)\Big]_{x=0}^{x=\infty} = 2(-0 + 2\cdot 0) - 2\left(-\infty + 2\cdot\frac{\pi}{2}\right) = \infty - 2\pi = \infty.$$

Perhaps this should be done with $\lim_{x\to\infty}(-x + 2\arctan x)$.

(15a) Find the area between $y=x^2$ and $y=2x$.

First graph
$y=2x$ and
$y=x^2$

The curves intersect at $(0,0)$ and $(2,4)$

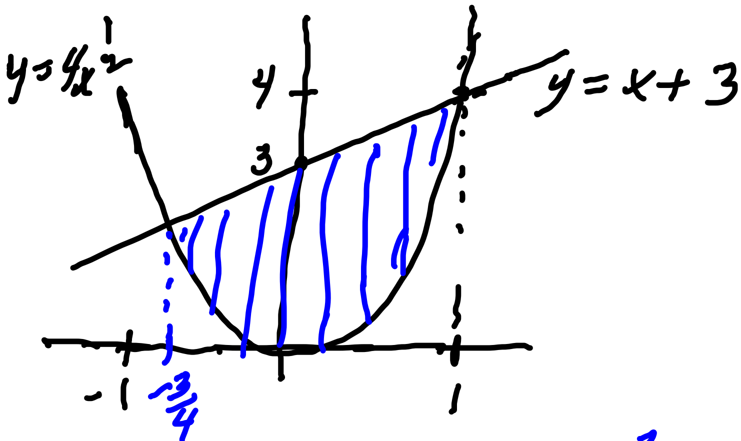


$$\begin{pmatrix}\text{Area between} \\ y=x^2 \text{ and } y=2x\end{pmatrix} = \int_{x=0}^{x=2} (2x-x^2)\,dx = x^2-\tfrac{1}{3}x^3\Big]_{x=0}^{x=2}$$

$$=(4-\tfrac{1}{3}8)-(0-0)=\tfrac{4}{3}.$$

(15b) Find the area between $y=4x^2$ and $y=x+3$.

First graph
$y=4x^2$ and $y=x+3$



Intersection points at
$4x^2=x+3$.

$4x^2-x-3=0$
$(x-1)(4x+3)=0$. So $x=1$ or $x=\frac{-3}{4}$

$$\begin{pmatrix}\text{Area between} \\ y=4x^2 \text{ and} \\ y=x+3\end{pmatrix} = \int_{x=-3/4}^{x=1} (x+3-4x^2)\,dx = \tfrac{1}{2}x^2+3x-\tfrac{4}{3}x^3\Big]_{x=-\frac{3}{4}}^{x=1}$$

$$=\left(\tfrac{1}{2}1^2+3\cdot 1-\tfrac{4}{3}1^3\right)-\left(\tfrac{1}{2}\left(\tfrac{-3}{4}\right)^2+3\cdot\tfrac{4}{3}-\tfrac{4}{3}\left(\tfrac{4}{3}\right)^3\right)$$

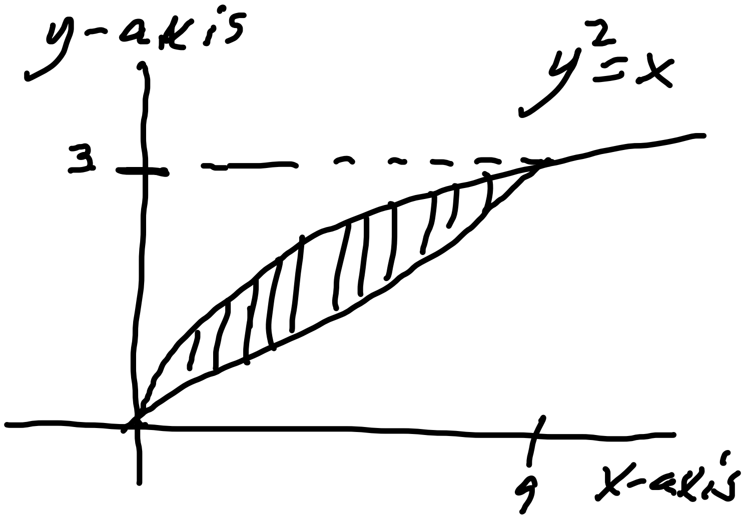
$$=\tfrac{1}{2}+3-\tfrac{4}{3}-\frac{3^2}{2^5}-4+\frac{2^8}{3^4}=\tfrac{-1}{2}-\tfrac{4}{3}-\frac{3^2}{2^5}+\frac{2^8}{3^4}.$$

(15a) Find the area between $y^2=x$ and $y=\frac{1}{3}x$.

First graph $y^2=x$ and $y=\frac{1}{3}x$.

The intersection points are at $x=y^2=\left(\frac{1}{3}x\right)^2=\frac{1}{9}x^2$

So $x=0$ or $x=9$.



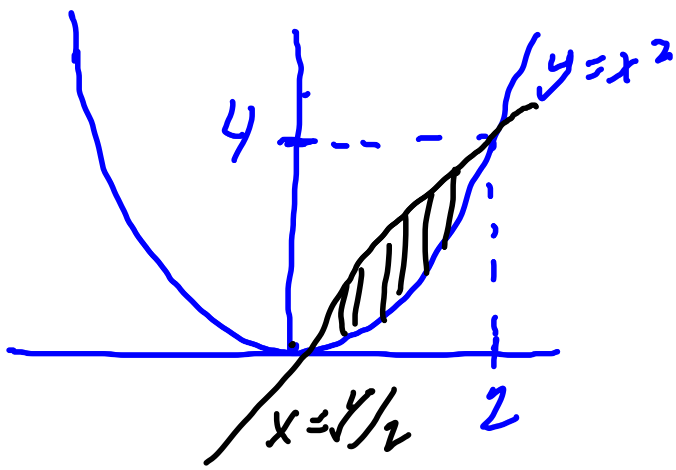
$$\left(\begin{array}{l}\text{Area between } y^2=x\\ \text{and } y=\frac{1}{3}x\end{array}\right)=\int_{x=0}^{x=9}\left(\sqrt{x}-\tfrac{1}{3}x\right)dx=\left.\frac{2x^{3/2}}{3}-\frac{1}{6}x^2\right]_{x=0}^{x=9}$$

$$=\left(\tfrac{2}{3}9^{3/2}-\tfrac{1}{6}9^2\right)-(0-0)=\tfrac{2}{3}3^3-\tfrac{27}{2}=27\left(\tfrac{2}{3}-\tfrac{1}{2}\right)=27\cdot\tfrac{1}{6}=\tfrac{9}{2}.$$

(16a) Find the area between $x=\sqrt{y}$ and $y/2$.

First graph $x=\sqrt{y}$ and $x=y/2$

Intersection points at $\sqrt{y}=\frac{y}{2}$ so $y=\frac{y^2}{4}$ and $y=0$ and $y=4$.



$$\begin{pmatrix}\text{Area between } x=\sqrt{y} \\ \text{and } x=y/2\end{pmatrix} = \int_{x=0}^{x=2} (2x-x^2)\,dx = x^2-\tfrac{1}{3}x^3\Big]_{x=0}^{x=2}$$

$$= 4-\tfrac{1}{3}8-(0-0) = 4-\tfrac{8}{3} = \tfrac{4}{3}.$$

(16b) Find the area between $x=y^2$ and $x=3y$.

This is the same as (15c). Let us do it by integration over y instead of over x as we did for (15c).

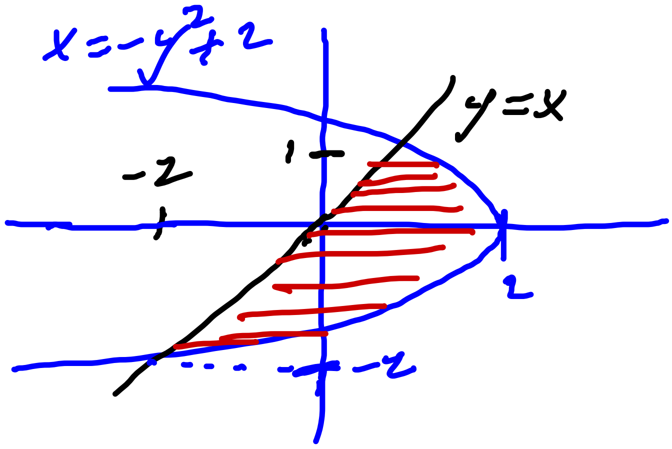
Intersection points are at $y^2=x=3y$, i.e. $y=0$ or $y=3$.

$$\begin{pmatrix}\text{Area between } x=y^2 \\ \text{and } x=3y\end{pmatrix} = \int_{y=0}^{y=3} (3y-y^2)\,dy = \tfrac{3}{2}y^2-\tfrac{1}{3}y^3\Big]_{y=0}^{y=3}$$

$$= \tfrac{3}{2}\cdot 3^2-\tfrac{1}{3}\cdot 3^3 = 27\left(\tfrac{1}{2}-\tfrac{1}{3}\right) = \tfrac{27}{6} = \tfrac{9}{2}.$$

(16c) Find the area between $x=-y^2+2$ and $y=x$.

First graph $x=-y^2+2$ and $y=x$. Intersection points at $y=x=-y^2+2$, i.e.
$0=y^2+y-2=(y+2)(y-1)$.
So $y=1$ and $y=-2$.



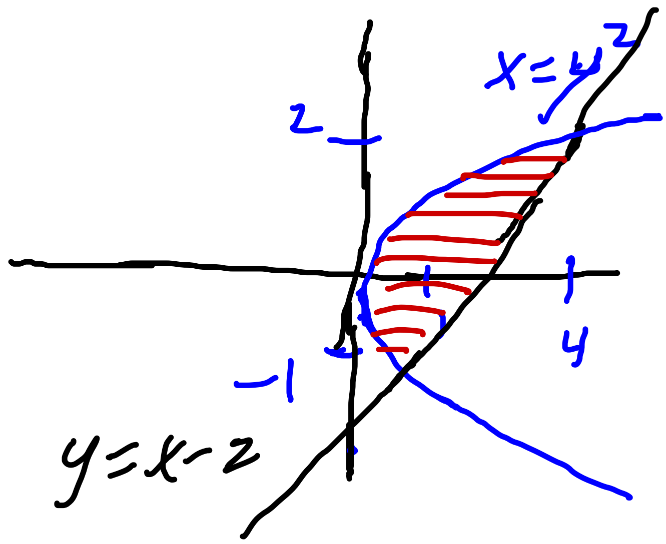
$$\begin{pmatrix}\text{Area between} \\ x=-y^2+2 \text{ and} \\ y=x\end{pmatrix} = \int_{y=-2}^{y=1} \left((-y^2+2)-y\right)dy$$

$$= \int_{y=-2}^{y=1} (-y^2 - y + 2)\,dy = -\tfrac{1}{3}y^3 - \tfrac{1}{2}y^2 + 2y \Big]_{y=-2}^{y=1} = \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(\tfrac{1}{3}\cdot 4 - \tfrac{1}{2}2^2 - 4\right)$$

$$= -\tfrac{1}{3} - \tfrac{1}{2} + 2 + \tfrac{4}{3} + 2 - 4 = \tfrac{1}{2}.$$

(17) Find the area between $y = x - 2$ and $x = y^2$ first by integrating with respect to x and then by integrating with respect to y.

First graph $y = x - 2$ and $x = y^2$. Intersection points are when $y + 2 = x = y^2$, $y^2 - y - 2 = 0$, so $0 = (y-2)(y+1)$, and $y = 2$, $x = 4$ or $y = -1$, $x = 1$.



$$\left(\begin{array}{l}\text{Area between } y = x-2 \\ \text{and } x = y^2\end{array}\right) = \int_{y=-1}^{y=2} (y + 2 - y^2)\,dy = \tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3 \Big]_{y=-1}^{y=2}$$

$$= \left(\tfrac{1}{2}\cdot 4 + 4 - \tfrac{8}{3}\right) - \left(\tfrac{1}{2} - 2 + \tfrac{1}{3}\right) = 2 + 4 - \tfrac{9}{3} - \tfrac{1}{2} + 2 = 8 - 3 - \tfrac{1}{2} = 4.5.$$

$$\left(\begin{array}{l}\text{Area between} \\ y = x-2 \text{ and } x = y^2\end{array}\right) = \int_{x=0}^{x=1} 2\sqrt{x}\,dx + \int_{x=1}^{x=4} (\sqrt{x} - (x-2))\,dx$$

$$= 2\cdot\tfrac{2}{3}x^{3/2}\Big]_{x=0}^{x=1} + \left(\tfrac{2}{3}x^{3/2} - \tfrac{1}{2}x^2 + 2x\right)\Big]_{x=1}^{x=4}$$

$$= 2\cdot\frac{2}{3}x^{3/2}\Big]_{x=0}^{x=1} + \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right)\Big]_{x=1}^{x=4}$$

$$= \left(\frac{4}{3} - 0\right) + \left(\frac{2}{3}\cdot 2^3 - \frac{1}{2}4^2 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{4}{3} + \frac{16}{3} - 8 + 8 - \frac{2}{3} + \frac{1}{2} - 2 = \frac{18}{3} + \frac{1}{2} - 2 = 6 + \frac{1}{2} - 2 = 4.5.$$