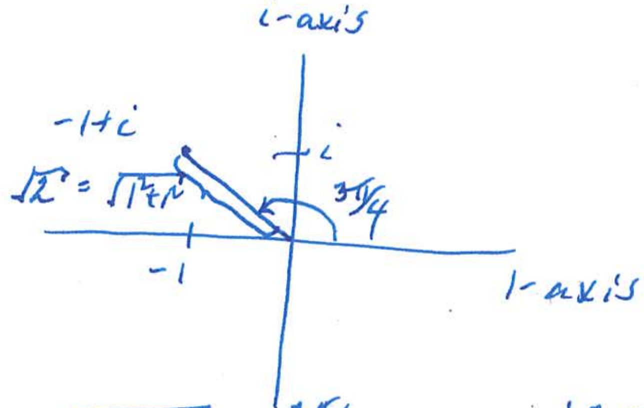
Calculus 2 Lecture 10

Example 3.1 Write $z=-1+i$ in polar form.

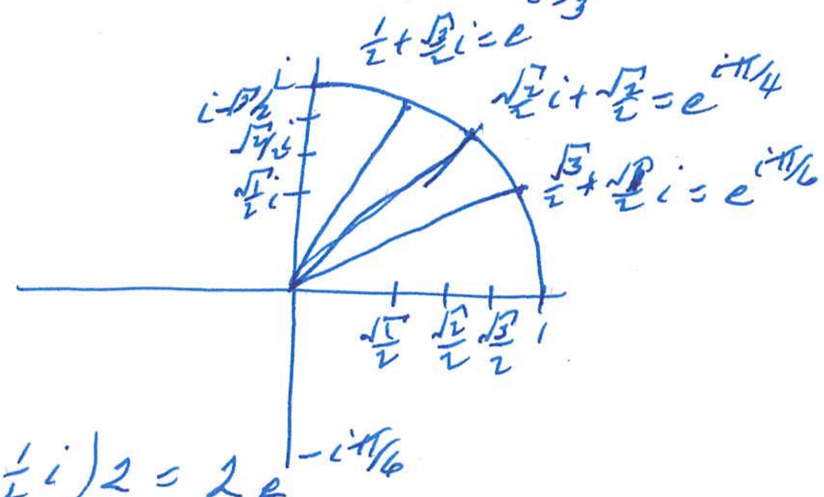
Solution



$$z=-1+i=\sqrt{1^2+1^2}\,e^{i\,3\pi/4}=\sqrt{2}\,e^{i\,3\pi/4}$$

Example 3.2 Simplify $(\sqrt{3}-i)(1+\sqrt{3}i)$ and $\dfrac{\sqrt{3}-i}{1+\sqrt{3}i}$.

Solution:



$$\sqrt{3}-i=\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)2=2e^{-i\pi/6}$$

$$1+\sqrt{3}i=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)2=2e^{i\pi/3}$$

So

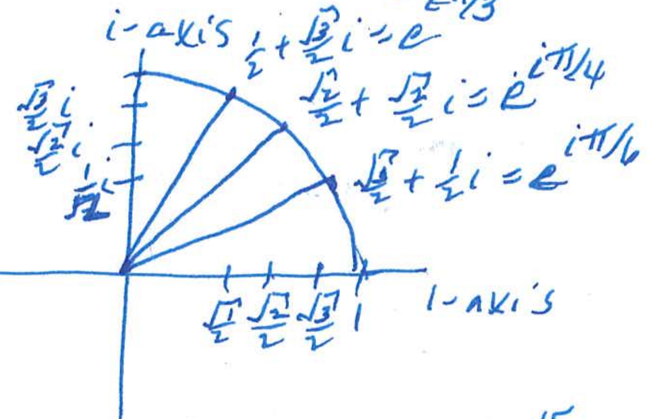
$$(\sqrt{3}-i)(1+\sqrt{3}i)=\left(2e^{-i\pi/6}\right)\left(2e^{i\pi/3}\right)=4e^{i\pi/6}=4\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$$=2\sqrt{3}+2i$$

$$\frac{\sqrt{3}-i}{1+\sqrt{3}i}=\frac{2e^{-i\pi/6}}{2e^{+i\pi/3}}=e^{-i\left(\frac{\pi}{6}+\frac{\pi}{3}\right)}=e^{-i\pi/2}=-i.$$

Example 3.3 Evaluate $(1+\sqrt{3}i)^{15}$

Solution:



$$(1+\sqrt{3}i)^{15} = \left(2\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)^{15} = 2^{15}\left(e^{i\pi/3}\right)^{15} = 2^{15}e^{i\frac{15}{3}\pi}$$

$$= 2^{15}e^{i5\pi} = 2^{15}e^{i4\pi}e^{i\pi} = 2^{15}e^{i\pi} = -2^{15}.$$

Example 3.4 Express $\sin^5\theta$ in terms of the functions $\sin(n\theta)$ for integers n.

Solution $\sin^5\theta = (\sin\theta)^5 = \left(\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\right)^5$

$$= \frac{1}{2^5 i^5}\left(e^{i\theta}-e^{-i\theta}\right)^5 = \frac{1}{2^5 i^4 i}\left(\begin{array}{l} e^{i5\theta} - 5e^{i4\theta}e^{-i\theta} + 10e^{i3\theta}e^{-i2\theta} \\ -10e^{i2\theta}e^{-i3\theta} + 5e^{i\theta}e^{-i4\theta} - e^{-i5\theta}\end{array}\right)$$

$$= \frac{1}{2^5 i}\cdot\frac{i}{i}\cdot\left(\begin{array}{l} e^{i5\theta} - e^{-i5\theta} - 5e^{i3\theta} + 5e^{-i3\theta} \\ + 10e^{i\theta} - 10e^{-i\theta}\end{array}\right)$$

$$= \frac{i}{-2^5}\left(i2\sin 5\theta - i2\cdot 5\sin(3\theta) + i2\cdot 10\sin\theta\right)$$

$$= \frac{-2}{-2^5}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$$

$$= \frac{1}{2^4}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$$

$$= \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta.$$

Example 3.5 Find $\frac{d^{56}}{dt^{56}}(e^{-t}\cos t)$.

Solution: $\frac{d^{56}}{dt^{56}}\left(e^{-t}(\cos t + i\sin t)\right)$

$= \frac{d^{56}}{dt^{56}}\left(e^{-t}e^{it}\right) = \frac{d^{56}}{dt^{56}}\left(e^{(1-i)t}\right)$

$= \frac{d^{55}}{dt^{55}}(1-i)e^{(1-i)t} = \frac{d^{54}}{dt^{54}}(1-i)^2 e^{(1-i)t}$

$= \frac{d^{53}}{dt^{53}}(1-i)^3 e^{(1-i)t} = \cdots = (1-i)^{56}e^{(1-i)t}$

$= \left(\sqrt{2}\, e^{-i\pi/4}\right)^{56} e^{t}(\cos t + i\sin t)$

$= 2^{56/2} e^{-i\pi\frac{56}{4}} e^{t}(\cos t + i\sin t)$

$= 2^{28} e^{-i\pi 14} e^{t}(\cos t + i\sin t)$

$= 2^{28}e^{t}\cos t + i2^{28}e^{t}\sin t.$

So $\frac{d^{56}}{dt^{56}}e^{-t}\cos t = 2^{28}e^{t}\cos t$

<u>Example 3.6</u> Evaluate $\int e^{3x} \sin(2x)\, dx$

<u>Solution</u>

$$i\int e^{3x} \sin 2x \, dx + \int e^{3x} \cos 2x \, dx$$

$$= \int e^{3x} e^{i2x} dx = \int e^{(3+2i)x} dx = \frac{1}{3+2i} e^{(3+2i)x} + c_1 + ic_2$$

$$= \frac{1}{(3+2i)} \frac{(3-2i)}{(3-2i)} \cdot e^{3x} (\cos 2x + i \sin 2x) + c_1 + ic_2$$

$$= \frac{(3-2i)}{3^2+2^2} e^{3x} (\cos 2x + i \sin 2x) + c_1 + ic_2$$

$$\cancel{= \frac{3}{13} e^{3x} \cos 2x - \frac{2}{13} i^2 e^{3x} \sin 2x + c_1 + ic_2}$$

$$\cancel{= \frac{3}{13} e^{3x} \cos 2x + c_1 + 2}$$

$$= \frac{3}{13} e^{3x} \cos 2x + \frac{2}{13} e^{3x} \sin 2x + c_1$$
$$+ i\left(\frac{3e^{3x}}{13} \sin 2x - \frac{2}{13} e^{3x} \cos 2x + c_2\right)$$

where c_1, c_2 are constants in $\mathbb{R}$.
So

$$\int e^{3x} \sin 2x \, dx = \frac{3}{13} e^{3x} \sin 2x - \frac{2}{13} e^{3x} \cos 2x + c_2$$

where c_2 is a constant in $\mathbb{R}$.