Calculus 2 Lecture 13

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Example 4.1 Evaluate $\int (6x^2+10)\sinh(x^3+5x-2)\,dx$

Solution:

$$\int (6x^2+10)\sinh(x^3+5x-2)\,dx$$

$$= \int 2(3x^2+5)\sinh(x^3+5x-2)\,dx$$

$$= 2\cosh(x^3+5x-2) + c, \text{ where } c \text{ is a constant,}$$

since

$$\frac{d}{dx}\left(2\cosh(x^3+5x-2)+c\right) = 2\sinh(x^3+5x-2)\cdot(3x^2+5)$$

$$= (6x^2+10)\sinh(x^3+5x-2).$$

Example 4.2 Evaluate $\int \frac{\operatorname{sech}^2(3x)}{10+2\tanh(3x)}\,dx$

Solution: Since

$$\frac{d}{dx}\log(10+2\tanh(3x)) = \frac{1}{10+2\tanh(3x)}\cdot 2\operatorname{sech}^2(3x)\cdot 3$$

$$= \frac{6\operatorname{sech}^2(3x)}{10+2\tanh(3x)}$$

then

$$\int \frac{\operatorname{sech}^2(3x)}{10+2\tanh(3x)}\,dx = \frac{1}{6}\log(10+2\tanh(3x)) + c,$$

where c is a constant. //

Example 4.3 Evaluate $\int \frac{1}{\sqrt{x^2+25}}\,dx$

Solution:

$$\int \frac{1}{\sqrt{x^2+25}}\,dx = \int \frac{1}{5\sqrt{\frac{x^2}{25}+1}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{x}{5}\right)^2+1}}\,dx$$

Let $\frac{x}{5} = \sinh u$. Then $x = 5\sinh u$ and $\frac{dx}{du} = 5\cosh u$.

$$\frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{x}{5}\right)^2+1}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{\sinh^2 u+1}}\,\frac{dx}{du}\,du$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{\cosh^2 u}}\,5\cosh u\,du = \int du = u + c$$

$= \operatorname{arcsinh}\left(\frac{x}{5}\right) + c$, where c is a constant.

Example 4.4 Evaluate $\int \frac{1}{x^2+2}\,dx$

Solution: $\int \frac{1}{x^2+2}\,dx = \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\,dx = \frac{1}{2}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx$

Let $\frac{x}{\sqrt{2}} = \tan u$ so that $\frac{dx}{du} = \sqrt{2}\sec^2 u$. Then

$$\frac{1}{2}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx = \frac{1}{2}\int \frac{1}{\tan^2 u+1}\,\frac{dx}{du}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sec^2 u}\,\sqrt{2}\sec^2 u\,du = \frac{1}{\sqrt{2}}\int du$$

$= \frac{1}{\sqrt{2}}u + c = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c$, where c is a constant.

Example 4.5 Evaluate $\int \sqrt{9-4x^2}\,dx$

Solution: $\int \sqrt{9-4x^2}\,dx = \int 3\sqrt{1-\frac{4}{9}x^2}\,dx$

$= 3\int \sqrt{1-\left(\frac{2}{3}x\right)^2}\,dx$. Let $\frac{2}{3}x = \sin u$.

Then $x = \frac{3}{2}\sin u$ and $\frac{dx}{du} = \frac{3}{2}\cos u$.

$3\int \sqrt{1-\left(\frac{2}{3}x\right)^2}\,dx = 3\int \sqrt{1-\sin^2 u}\,\frac{dx}{du}\,du$

$= 3\int \sqrt{\cos^2 u}\,\frac{3}{2}\cos u\,du = \frac{9}{2}\int \cos^2 u\,du$

Since $\cos 2u = \cos^2 u - \sin^2 u = \cos^2 u - (1-\cos^2 u)$

$= 2\cos^2 u - 1,$

then $\cos 2u + 1 = 2\cos^2 u$ and $\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$.

So $\frac{9}{2}\int \cos^2 u\,du = \frac{9}{2}\int\left(\frac{1}{2} + \frac{1}{2}\cos 2u\right)du$

$= \frac{9}{2}\left(\frac{1}{2}u + \frac{1}{2}\sin(2u)\,\frac{1}{2}\right) + c$

$= \frac{9}{4}u + \frac{9}{8}\sin u\cos u + c$

$= \frac{9}{4}\arcsin\left(\frac{2}{3}x\right) + \frac{9}{8}\cdot\frac{2}{3}x\cdot\sqrt{1-\left(\frac{2}{3}x\right)^2} + c$

$= \frac{9}{4}\arcsin\left(\frac{2}{3}x\right) + \frac{18}{24}x\sqrt{1-\frac{4}{9}x^2} + c$, where c is a constant.

Example 4.6 Evaluate $\int (x^2-1)^{3/2}\,dx$.

Solution: Let $x = \cosh u$. Then $\frac{dx}{du} = \sinh u$ and $x^2 - 1 = \cosh^2 u - 1 = \sinh^2 u$.

$$\int (x^2-1)^{3/2}dx = \int (x^2-1)^{3/2}\frac{dx}{du}\,du = \int (\sinh^2 u)^{3/2} \sinh u\, du$$

$$= \int \sinh^4 u\, du = \int \left(\frac{1}{2}(e^u - e^{-u})\right)^4 du$$

$$= \int \frac{1}{2^4}\left(e^{4u} - 4e^{2u} + 6 - 4e^{-2u} + e^{-4u}\right) du$$

$$= \frac{1}{2^4}\left(\frac{e^{4u}}{4} - \frac{4e^{2u}}{2} + 6u - \frac{4e^{-2u}}{-2} + \frac{e^{-4u}}{-4}\right) + c$$

$$= \frac{1}{2^4}\left(\frac{1}{4}(e^{4u} - e^{-4u}) - 2(e^{2u} - e^{-2u}) + 6u\right) + c$$

$$= \frac{1}{2^4}\left(\frac{1}{2}\sinh(4u) - 4\sinh 2u + 6u\right) + c$$

$$= \frac{1}{2^5}\sinh(2u+2u) - \frac{4}{2^4}\sinh(u+u) + \frac{6}{2^4}u + c$$

$$= \frac{1}{2^5}2\sinh(2u)\cosh(2u) - \frac{1}{2^2}2\sinh u\cosh u + \frac{3}{2^3}u + c$$

$$= \frac{1}{2^4}2\sinh u\cosh u(\cosh^2 u + \sinh^2 u) - \frac{1}{2}\sinh u\cosh u + \frac{3}{8}u + c$$

$$= \frac{1}{2^3}\sinh u\cosh^3 u + \frac{1}{2^3}\sinh^3 u\cosh u - \frac{1}{2}\sinh u\cosh u + \frac{3}{8}u + c$$

$$= \frac{1}{8}\sqrt{\cosh^2 u - 1}\,\cosh^3 u + \frac{1}{8}\left(\sqrt{\cosh^2 u - 1}\right)^3 \cosh u$$
$$- \frac{1}{2}\sqrt{\cosh^2 u - 1}\,\cosh u + \frac{3}{8} u + c$$

$$= \frac{1}{8}\sqrt{x^2-1}\,x^3 + \frac{1}{8}(x^2-1)^{3/2}x - \frac{1}{2}\sqrt{x^2-1}\,x$$
$$+ \frac{3}{8}\operatorname{arccosh} x + c$$

$$= (x^2-1)^{\frac{1}{2}}\left(\frac{1}{8}x^3 + \frac{1}{8}x(x^2-1) - \frac{1}{2}x\right)$$
$$+ \frac{3}{8}\operatorname{arccosh} x + c$$

$$= (x^2-1)^{\frac{1}{2}}\left(\frac{1}{4}x^3 - \frac{5}{8}x\right) + \frac{3}{8}\operatorname{arccosh} x + c$$

$$= \frac{1}{8}(x^2-1)^{\frac{1}{2}}(2x^3 - 5x) + \frac{3}{8}\operatorname{arccosh} x + c,$$

where c is a constant.

Example 4.7 In the course of the previous question we determined

$$\int \sinh 4\theta \, d\theta = \frac{1}{2^5}\sinh(4\theta) - \frac{1}{2^2}\sinh 2\theta + \frac{3}{8}\theta + c$$

where c is a constant.