Calculus 2 Lect 15 continued

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①

Example 4.12 Evaluate $\int \frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}\,dx$.

Solution If we find $A_1, A_2, A_3, A_4, A_5, A_6 \in \mathbb{C}$ so that

$$\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)} = \frac{2x^4+3x^2}{((x-i)(x+i))^2(x-\sqrt{2}i)(x+\sqrt{2}i)}$$

$$= \frac{2x^4+3x^2}{(x-i)^2(x+i)^2(x-\sqrt{2}i)(x+\sqrt{2}i)}$$

$$= \frac{A_1}{x-i} + \frac{A_2}{(x-i)^2} + \frac{A_3}{x+i} + \frac{A_4}{(x+i)^2} + \frac{A_5}{x-\sqrt{2}i} + \frac{A_6}{x+\sqrt{2}i}$$

then

$$\int \frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}$$

$$= \int \left(\frac{A_1}{x-i} + A_2(x-i)^{-2} + \frac{A_3}{x+i} + A_4(x+i)^{-2} + \frac{A_5}{x-\sqrt{2}i} + \frac{A_6}{x+\sqrt{2}i} \right) dx$$

$$= A_1\log(x-i) - A_2(x-i)^{-1} + A_3\log(x+i) - A_4(x+i)^{-1}$$
$$+ A_5\log(x-\sqrt{2}i) + A_6\log(x+\sqrt{2}i) + c,$$

where c is a constant.

Find A_5:

$$\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}(x-\sqrt{2}i) = \left(\frac{A_1}{x-i} + \frac{A_2}{(x-i)^2} + \frac{A_3}{x+i} + \frac{A_4}{(x+i)^2} + \frac{A_6}{x+\sqrt{2}i}\right)(x-\sqrt{2}i) + A_5$$

so that

$$\frac{2x^4+3x^2}{(x^2+1)^2(x+\sqrt{2}i)} = \left(\frac{A_1}{x-i} + \dots + \frac{A_6}{x+\sqrt{2}i}\right)(x-\sqrt{2}i) + A_5$$

and, by plugging in $x=\sqrt{2}i$ this gives

$$\frac{2(\sqrt{2}i)^4+3(\sqrt{2}i)^2}{((\sqrt{2}i)^2+1)^2(\sqrt{2}i+\sqrt{2}i)} = \left(\frac{A}{\sqrt{2}-i} + \dots + \frac{A_6}{\sqrt{2}i+\sqrt{2}i}\right)\cdot 0 + A_5$$

so that

$$A_5 = \frac{2\cdot 2^2 i^4 + 3\cdot 2(-1)}{(2(-1)+1)^2\, 2\sqrt{2}i} = \frac{8-6}{(-1)^2 2\sqrt{2}i} = \frac{1}{\sqrt{2}i} = \frac{-i}{\sqrt{2}}.$$

Since

$$\frac{A_1}{(x-i)}+\frac{A_2}{(x-i)^2}+\frac{A_3}{(x+i)}+\frac{A_4}{(x+i)^2}+\frac{A_5}{x-\sqrt{2}i}+\frac{A_6}{x+\sqrt{2}i}$$

$$=\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}=\overline{\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}}$$

$$=\frac{\overline{A_1}}{x+i}+\frac{\overline{A_2}}{(x+i)^2}+\frac{\overline{A_3}}{x-i}+\frac{\overline{A_4}}{(x-i)^2}+\frac{\overline{A_5}}{x+\sqrt{2}i}+\frac{\overline{A_6}}{x-\sqrt{2}i}$$

then $\overline{A_1}=A_3$, $\overline{A_2}=A_4$, $\overline{A_5}=A_6$.

So

$$A_6=\overline{A_5}=\overline{\frac{-1}{\sqrt{2}}i}=\frac{1}{\sqrt{2}}i.$$

Now

$$\frac{A_5}{x-\sqrt{2}i}+\frac{A_6}{x+\sqrt{2}i}=\frac{\frac{-1}{\sqrt{2}}i}{x-\sqrt{2}i}+\frac{\frac{1}{\sqrt{2}}i}{x+\sqrt{2}i}$$

$$=\frac{\frac{-1}{\sqrt{2}}ix-i^2+\frac{1}{\sqrt{2}}ix-i^2}{(x-\sqrt{2}i)(x+\sqrt{2}i)}=\frac{1+1}{x^2+2}=\frac{2}{x^2+2}.$$

So

$$\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}=\frac{A_1}{x-i}+\frac{A_2}{(x-i)^2}+\frac{\overline{A_1}}{x+i}+\frac{\overline{A_2}}{(x+i)^2}$$

$$+\frac{2}{x^2+2}$$

$\therefore$

$$\frac{A_1}{x-i}+\frac{A_2}{(x-i)^2}+\frac{\overline{A_1}}{x+i}+\frac{\overline{A_2}}{(x+i)^2}$$

$$=\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}-\frac{2}{x^2+2}$$

$$=\frac{2x^4+3x^2-2(x^2+1)^2}{(x^2+1)^2(x^2+2)}$$

$$=\frac{2x^4+3x^2-2x^4-4x^2-2}{(x^2+1)^2(x^2+2)}$$

$$=\frac{-x^2-2}{(x^2+1)^2(x^2+2)}=\frac{-1}{(x^2+1)^2}$$

$\therefore$

$$\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}=\frac{-1}{(x^2+1)^2}+\frac{2}{x^2+2}.$$

$$=\frac{-1}{(x^2+1)^2}+\frac{1}{\frac{x^2}{2}+1}$$

$$=\frac{-1}{(x^2+1)^2}+\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}$$

$$\frac{A_2}{(x-i)^2}+\frac{\bar{A}_2}{(x+i)^2} = \frac{A_2(x^2+2ix+i^2)+\bar{A}_2(x^2-2ix+i^2)}{(x^2+1)^2}$$

$$= \frac{(A_2+\bar{A}_2)x^2+(A_2-\bar{A}_2)2ix-(A_2+\bar{A}_2)}{(x^2+1)^2}$$

Then

$$A_2 = \frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}(x-i)^2\Big|_{x=i}$$

$$= \frac{2x^4+3x^2}{(x+i)^2(x^2+2)}\Big|_{x=i} = \frac{2i^4+3i^2}{(2i)^2(-1+2)} = \frac{2-3}{(-4)\cdot 1}$$

$$= \frac{-1}{-4} = \frac{1}{4}.$$

So

$$\frac{\frac{1}{4}}{(x-i)^2}+\frac{\frac{1}{4}}{(x+i)^2} = \frac{\frac{1}{2}x^2-\frac{1}{2}}{(x^2+1)^2} = \frac{\frac{1}{2}(x^2-1)}{(x^2+1)^2}$$

and

$$\frac{-1}{(x^2+1)^2} - \frac{(\frac{1}{2}x^2-\frac{1}{2})}{(x^2+1)^2} = \frac{-\frac{1}{2}x^2-\frac{1}{2}}{(x^2+1)^2} = \frac{-\frac{1}{2}}{x^2+1}$$

So

$$\frac{-\frac{1}{2}}{x^2+1} = \frac{A_1}{x-i}+\frac{\bar{A}_1}{x+i} = \frac{(A_1+\bar{A}_1)x+i(A_1-\bar{A}_1)}{x^2+1}$$

So $2\,\mathrm{Re}(A_1) = -\frac{1}{2}$ and $2\,\mathrm{Im}(A_1) = 0$

So $A_1 = -\frac{1}{4}$.

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Remark

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) = \frac{1}{i}\sinh(ix) = -i\sinh(ix)$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) = \cosh(ix)$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{2i}(e^{ix} - e^{-ix})}{\frac{1}{2}(e^{ix} + e^{-ix})} = -i\tanh(ix)$$

Then

$$\tan\left(\frac{x}{a}\right) = -i\tanh\left(\frac{ix}{a}\right) = -i\,\frac{1}{2}\log\left(\frac{1+i\frac{x}{a}}{1-i\frac{x}{a}}\right)$$

$$= \frac{1}{2i}\log\left(\frac{a+ix}{a-ix}\right) = \frac{1}{2i}\log\left(\frac{ai-x}{ai+x}\right).$$

$$= \frac{1}{2i}\log(ai-x) - \frac{1}{2i}\log(ai+x).$$

$$= -\frac{i}{2}\log(ai-x) + \frac{i}{2}\log(ai+x)$$

$$= \frac{i}{2}\left(\log(ai+x) - \log(ai-x)\right).$$