Calculus 2 Lecture 16 02.09.2019
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Example 5.1 $3\frac{d^4y}{dx^4} = \left(\frac{dy}{dx}\right)^2 + 2x^2y$ is order 4.

Example 5.2 Verify that $y = x^2 + \frac{2}{x}$ is a solution of $\frac{dy}{dx} + \frac{y}{x} = 3x$.

Solution: Let $y = x^2 + 2x^{-1}$. Then

$$\frac{dy}{dx} + \frac{y}{x} = 2x + 2(-1)x^{-2} + \frac{y}{x}$$
$$= 2x - \frac{2}{x^2} + \frac{1}{x}\left(x^2 + \frac{2}{x}\right)$$
$$= 2x - \frac{2}{x^2} + x + \frac{2}{x^2} = 3x. //$$

Example 5.3 Solve $\frac{dy}{dx} = x^2$.

Solution: If $y = \frac{1}{3}x^3$ then $\frac{dy}{dx} = x^2$.
If $y = \frac{1}{3}x^3 + c$, where c is a constant, then $\frac{dy}{dx} = x^2$. //

Example 5.4 Solve $\frac{dy}{dx} = x^2$ with $y(0) = 2$.

Solution If $y = \frac{1}{3}x^3 + c$ then $\frac{dy}{dx} = x^2$.
If $2 = y(0) = \frac{1}{3}\cdot 0^3 + c$ then $c = 2$.
So $y = \frac{1}{3}x^3 + 2$. //

Example 5.5 Solve $\frac{dy}{dx} = \frac{y}{1+x}$

Solution
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x}.$$

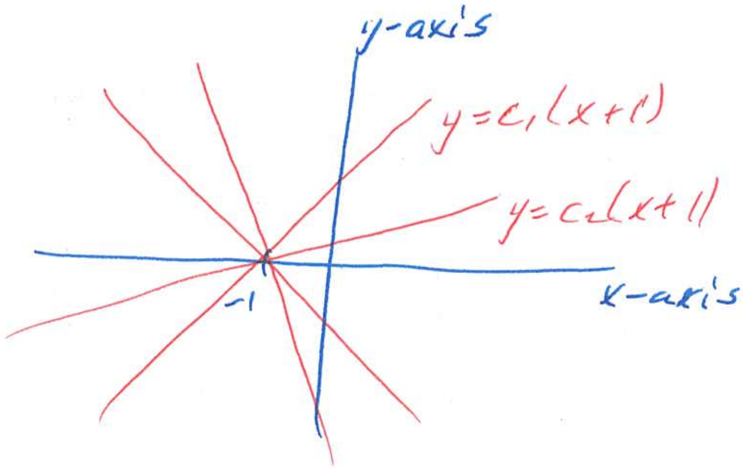
So
$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{1+x}\,dx.$$

So $\log y = \log(1+x) + c.$

So $y = e^{\log(1+x)+c} = e^{\log(1+x)} \cdot e^{c}$

$= C(1+x)$, where C is a constant.

The graph of $y = Cx + C = C(x+1)$, is a line of slope C which crosses the x-axis at $x = -1$.



Example 5.6 Solve $\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}}$ if $y(0)=3$.

Solution: $$\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}}.$$

So $$2y\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

So $$\int 2y\frac{dy}{dx}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

Letting $x = \sin\theta$ then $\frac{dx}{d\theta} = \cos\theta$ and

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\,\frac{dx}{d\theta}\,d\theta = \int \frac{1}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$$

$$= \int d\theta = \theta + c = \arccos x + c$$

So $$\int 2y\frac{dy}{dx}\,dx = \arccos x + c$$

So $$y^2 = \arccos x + c.$$

If $y(0)=3$ then $3^2 = \arccos(0) + c$

$= \frac{\pi}{2} + c$ and $c = 9 - \frac{\pi}{2}$.

So $y^2 = \arccos x + \left(9 - \frac{\pi}{2}\right)$ and

$$y = \sqrt{\arccos x + \left(9 - \frac{\pi}{2}\right)}.$$

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Example 5.7 Solve $x\frac{dy}{dx}+y=e^x$.

Solution Since $\frac{d}{dx}(xy)=x\frac{dy}{dx}+y$ then

$\frac{d}{dx}(xy)=e^x$. So $\int \frac{d}{dx}(xy)\,dx=\int e^x\,dx$

So $xy=e^x+c$. So $y=\frac{1}{x}e^x+c\cdot\frac{1}{x}$,

where c is a constant. //.

Example 5.8 Solve $\frac{dy}{dx}+\frac{y}{x}=\sin x$.

Solution: $\frac{dy}{dx}+\frac{y}{x}=\sin x$.

So
$$x\frac{dy}{dx}+y=x\sin x$$

So
$$\frac{d}{dx}(xy)=x\sin x.$$

So $\int\frac{d}{dx}(xy)\,dx=\int x\sin x\,dx$.

So
$$xy=\int x\sin x\,dx=\int\left(\frac{d}{dx}(-x\cos x)+\cos x\right)dx$$
$$=-x\cos x+\sin x+c$$

So $y=-\cos x+\frac{1}{x}\sin x+\frac{c}{x}$, where c is a constant.

Example 5.9 Solve $\frac{1}{2}\frac{dy}{dx} - xy = x$ if $y(0) = -3$.

Solution: $\frac{1}{2}\frac{dy}{dx} - xy = x$.

So $\frac{1}{2}\frac{dy}{dx} = x + xy = x(1+y)$.

So $\frac{1}{2} \cdot \frac{1}{(1+y)} \cdot \frac{dy}{dx} = x$.

So $\int \frac{1}{2} \frac{1}{(1+y)} \frac{dy}{dx}\, dx = \int x\, dx$.

So $\frac{1}{2} \log(1+y) = \frac{1}{2}x^2 + c_1$, where c_1 is a constant.

So $\log(1+y) = x^2 + c_2$, where c_2 is a constant.

So $1+y = e^{x^2+c_2} = e^{x^2} e^{c_2} = c_3 e^{x^2}$.

So $y = c_3 e^{x^2} - 1$, where c_3 is a constant.