Example 5.15 Effluent (pollutant concentration $2g/m^3$) flows into a pond (volume $1000 m^3$, initially 100 g pollutant) at a rate of $10 m^3/min$. The pollutant mixes quickly and uniformly with pond water and flows out of pond at a rate of $10 m^3/min$. Find the concentration of pollutant in the pond at any time.

Solution:

$x = x(t)$ = amount of pollutant in pond.

$x(0) = 100$

C = concentration of pollutant in pond

$= \frac{x}{V} = \frac{x}{1000}$, since V = Volume of pond $= 1000$.

$\frac{dx}{dt}$ = rate pollutant in − rate pollutant out

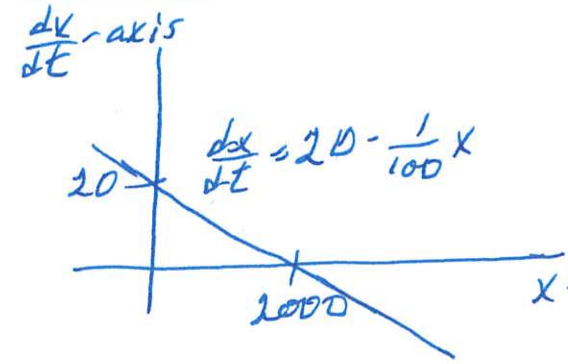
$= 2 \cdot 10 - C \cdot 10 = 20 - \frac{x}{V} \cdot 10$

$= 20 - \frac{x}{1000} \cdot 10 = 20 - \frac{x}{100}$

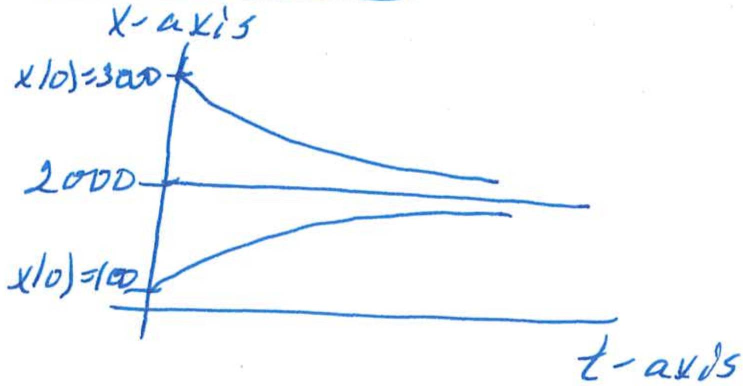
So

$$\frac{dx}{dt} = 20 - \frac{1}{100}x \quad \text{and} \quad x(0) = 100$$

Phase plot



Solution plot



Exact solution $\left(\dfrac{1}{20-\frac{1}{100}x}\right)\dfrac{dx}{dt} = 1$

So $\displaystyle\int \left(\frac{1}{20-\frac{1}{100}x}\right)\frac{dx}{dt}\,dt = \int dt.$

So $-100\cdot\log\left(20-\frac{1}{100}x\right) = t + c_1.$

So $\log\left(20-\frac{1}{100}x\right) = -\frac{1}{100}t + \frac{c_1}{100}.$

So $20-\frac{1}{100}x = e^{-\frac{1}{100}t - \frac{c_1}{100}} = C\cdot e^{-\frac{1}{100}t}.$

So $\frac{1}{100}x = 20 - Ce^{-\frac{1}{100}t}$

So $x = 2000 - 100Ce^{-\frac{1}{100}t}$, where C is a constant.

If $x(0)=100$ then $100 = 2000 - 100Ce^{0}$
$= 2000 - 100C$

and $C=19$. So

$$x = \underbrace{2000}_{\text{steady state term}} - \underbrace{1900e^{-\frac{1}{100}t}}_{\text{transient term}}.$$

Example 5.16 Find the concentration of pollutant in the pond if the input flow rate is decreased to $5m^3/min$.

Solution: In this case the volume of the lake is decreasing,

$$V = 1000 - 5t.$$

So $\frac{dx}{dt} = 2\cdot 5 - C\cdot 10 = 10 - \frac{x}{V}\cdot 10 = 10 - \frac{10}{1000-5t}x.$

So $\frac{dx}{dt} + x\left(\frac{10}{1000-5t}\right) = 10.$

We want I so that

$$\frac{dx}{dt}I + x\left(\frac{10}{1000-5t}\right)I = \frac{d(xI)}{dt} = \frac{dx}{dt}I + x\frac{dI}{dt}.$$

So $\frac{dI}{dt} = \frac{10}{1000-5t}I$ and $\frac{1}{I}\frac{dI}{dt} = \frac{10}{1000-5t}$

So $\log I = \frac{10}{-5}\log(1000-5t) = \log((1000-5t)^{-2}) + c,$

So $I = C(1000-5t)^{-2}.$

So $\frac{d(xI)}{dt} = (\text{lefthandside})I = (\text{righthandside})I$

$= 10\cdot I = 10C(1000-5t)^{-2}.$

So $xI = \frac{10C}{(-1)(-5)}(1000-5t)^{-1} + c_2$

So $Cx(1000-5t)^{-2} = 2C(1000-5t)^{-1} + c_2$

So $x = 2(1000-5t) + \frac{c_2}{C}(1000-5t)^2$

$= 2(1000-5t) + c_3(1000-5t)^2,$

where c_3 is a constant.

~~Since $x = \frac{C}{V}$ and $V = 1000-5t$ then~~

~~concentration is~~

$\cancel{C = Vx = (1000-5t)(}$

Since $C = \frac{x}{V}$ and $V = 1000-5t$ then

concentration is

$$C = \frac{x}{1000-5t} = 2 + c_3(1000-5t).$$

If $x(0) = 1000$ then $100 = 2\cdot(1000-0) + c_3(1000-0)^2$

and $c_3 = \frac{100-2000}{1000^2} = -1900\cdot 10^{-6} = -1.9\cdot 10^{-3}$

So $C = 2 - 1.9(1000-5t)\cdot 10^{-3}$

$= 2 - \frac{1900}{1000} + \frac{9.5}{1000}t = 2 - 1.9 + .095t = 0.1 + .095t$