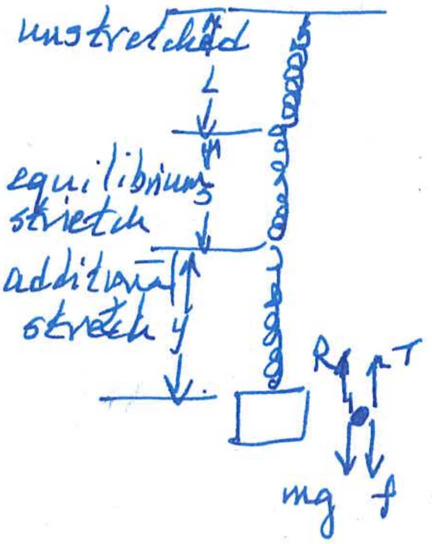
Calculus 2 Lecture 24

Hanging spring

$$my'' + \beta y' + ky = f$$

- m: mass
- β: damping constant
- k: spring constant
- f: external downward force



Forces

gravity: mg

restoring force: $T = k(s+y)$

damping force: $R = \beta y'$

external downward force: f

Newtons Law: $ma = F$ gives

$$my'' = mg - T - R + f$$
$$= mg - k(s+y) - \beta y' + f$$

and

$$mg = ks$$

since mg and T balance at equilibrium.

Example 6.10 Let $m = \frac{40}{49}$ kg, $s = 0.2$m

$y'(0) = 3$m/s and $y(0) = 0$.

Find y if

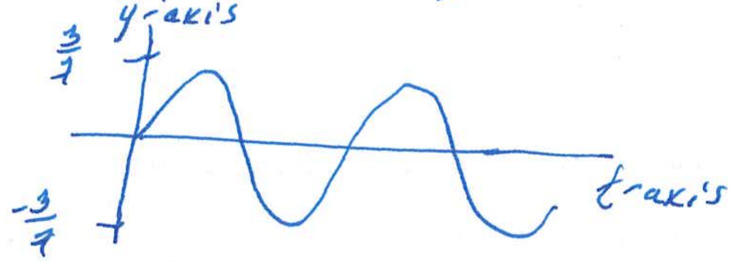
(a) $\beta = 0$, (b) $\beta = \frac{160}{49}$, (c) $\beta = \frac{80}{7}$, (d) $\beta = \frac{2000}{49}$.

Answer $k = \frac{mg}{s} = \frac{9.8m}{0.2} = \frac{9.8 \cdot \frac{40}{49}}{0.2} = 40.$

So $\frac{40}{49} y'' + \beta y' + 40y = 0.$

(a) $\beta = 0$ (so $\beta^2 - 4 \cdot \frac{40}{49} \cdot 40 < 0$)

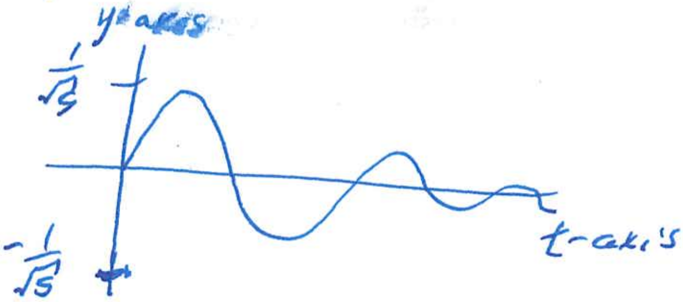
$y = \frac{3}{7} \sin(7t)$



④ simple harmonic motion

(b) $\beta = \frac{160}{49}$ (so $\beta^2 - 4 \cdot \frac{40}{49} \cdot 40 < 0$)

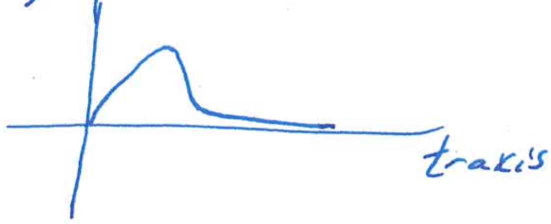
$y = \frac{1}{\sqrt{5}} e^{-2t} \sin(3\sqrt{5}\, t)$



④ weak damping

(c) $\beta = \frac{80}{7}$ (so $\beta^2 - 4 \cdot \frac{40}{49} \cdot 40 = 0$)

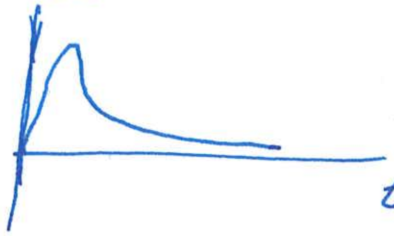
$y = 3te^{-7t}$



④ critical damping

(d) $\beta = \frac{2000}{49}$ $\left(50\beta^2 - 4\cdot\frac{40}{49}\cdot 40 > 0\right)$

$y = \frac{1}{16}\left(e^{-t} - e^{-49t}\right)$

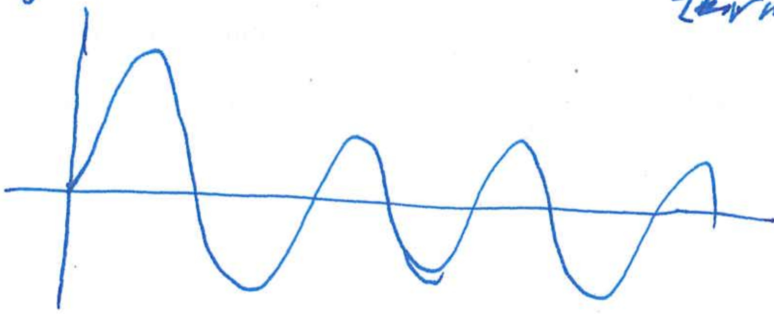


(4) strong damping

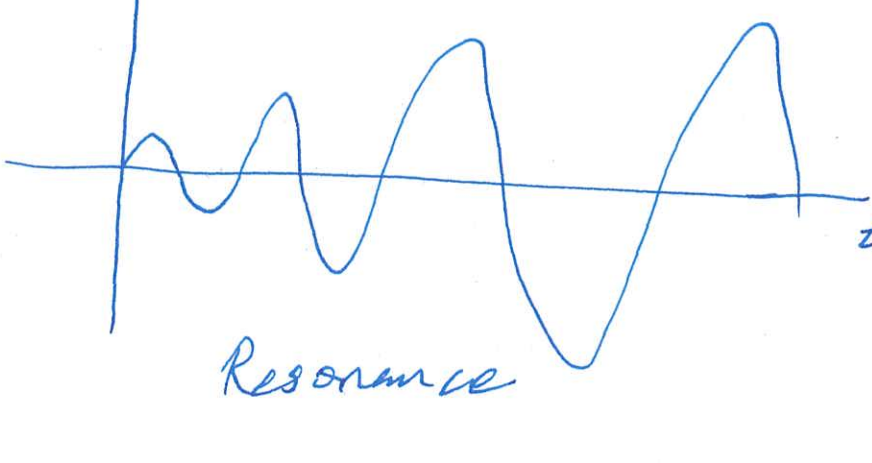
Example 6.11 $f = \frac{160}{7}\sin(7t)$.

(a) $\beta = \frac{80}{7}$ gives $y(t) = \left(\frac{2}{7} + 5t\right)e^{-7t} - \frac{2}{7}\cos(7t)$

transient term ↗ steady state term ↗



(b) $\beta = 0$ gives $y(t) = \frac{5}{7}\sin(7t) - 2t\cos(7t)$



Resonance

General solution

(a) If $\beta^2 - 4mk > 0$ then

$$y = \left(\frac{\lambda_2 y(0) - y'(0)}{\lambda_2 - \lambda_1}\right) e^{\lambda_1 t} + \left(\frac{\lambda_1 y(0) - y'(0)}{\lambda_1 - \lambda_2}\right) e^{\lambda_2 t}$$

with

$$\lambda_1 = \frac{-\beta + \sqrt{\beta^2 - 4mk}}{2m} \quad \text{and} \quad \lambda_2 = \frac{-\beta - \sqrt{\beta^2 - 4mk}}{2m}$$

(b) If $\beta^2 - 4mk = 0$ then

$$y = y(0) e^{-\frac{\beta}{2m}t} + \left(y'(0) + \frac{\beta}{2m} y(0)\right) t e^{-\frac{\beta}{2m}t}$$

(c) If $\beta^2 - 4mk < 0$ then

$$y = 2C e^{rt} \cos(\psi + \theta t)$$

where $r = \frac{-\beta}{2m}$, $\theta = \sqrt{4mk - \beta^2}$

$$C = \sqrt{y(0)^2 + \left(\frac{r y(0) - y'(0)}{\theta}\right)^2} \quad \text{and}$$

$$\psi = \arctan\left(\frac{1}{\theta}\left(r - \frac{y'(0)}{y(0)}\right)\right)$$