Calculus 2 Lecture 31

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Example 7.13 Let $f(x,y) = x\sin(x+2y)$.

Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial(x\sin(x+2y))}{\partial x} = x\cos(x+2y) + \sin(x+2y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\sin(x+2y)) = x\cos(x+2y)\cdot 2 = 2x\cos(x+2y)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(x\cos(x+2y) + \sin(x+2y))$$

$$= x(-\sin(x+2y)) + \cos(x+2y) + \cos(x+2y)$$

$$= -x\sin(x+2y) + 2\cos(x+2y).$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial(2x\cos(x+2y))}{\partial y} = 2x(-\sin(x+2y)\cdot 2)$$

$$= -4x\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2x\cos(x+2y)) = 2x(-\sin(x+2y)) + 2\cos(x+2y)$$

$$= -2x\sin(x+2y) + 2\cos(x+2y).$$

The Hessian matrix is

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2\cos(x+2y) - x\sin(x+2y) & -2x\sin(x+2y) + 2\cos(x+2y) \\ -2x\sin(x+2y) + 2\cos(x+2y) & -4x\sin(x+2y) \end{pmatrix}$$

Example 7.14 Let $z = x^2 - y^2$ and

$x = \sin t$ and $y = \cos t$.

Find $\frac{dz}{dt}$ at $t = \frac{\pi}{6}$.

Solution $\mathbb{R} \xrightarrow{(x,y)} \mathbb{R}^2 \xrightarrow{z} \mathbb{R}$

$$D_{(x,y)} = \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \text{ and } D_z = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}$$
$$= \begin{pmatrix} 2x & -2y \end{pmatrix}$$

Then

$$\frac{dz}{dt} = D_z D_{(x,y)} = \begin{pmatrix} 2x & -2y \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$= 2x\cos t + 2y\sin t = 2\sin t\cos t + 2\cos t\sin t$$
$$= 4\sin t\cos t.$$

So
$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{6}} = 4\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 4\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \sqrt{3}.$$

Example 7.15 Let $z = e^x \sinh y$ and
$x = st^2$ and $y = s^2 t$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution: $\mathbb{R}^2 \xrightarrow{(x,y)} \mathbb{R}^2 \xrightarrow{z} \mathbb{R}$

$$D(x,y) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} t^2 & 2st \\ 2st & s^2 \end{pmatrix}$$

$$Dz = \left(\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y} \right) = \left(e^x \sinh y \quad e^x \cosh y \right)$$

So

$$\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t} \right) = Dz \, D(x,y) = \left(e^x \sinh y \quad e^x \cosh y \right) \begin{pmatrix} t^2 & 2st \\ 2st & s^2 \end{pmatrix}$$

$$= \left(e^x \sinh y \, t^2 + e^x \cosh y \, 2st, \; e^x \sinh y \, 2st + e^x \cosh y \, s^2 \right)$$

So

$$\frac{\partial z}{\partial s} = e^{st^2} \cdot \sinh(s^2 t) \cdot t^2 + e^{st^2} \cosh(s^2 t) \cdot 2st$$

$$= t^2 e^{st^2} \sinh(s^2 t) + 2st e^{st^2} \cosh(s^2 t)$$

$$\frac{\partial z}{\partial t} = e^{st^2} \sinh(s^2 t) \cdot 2st + e^{st^2} \cosh(s^2 t) \cdot s^2$$

$$= 2st e^{st^2} \sinh(s^2 t) + s^2 e^{st^2} \cosh(s^2 t).$$

Example 7.16 Find the directional derivative of $f(x,y) = xe^y$ at $(2,0)$ in the direction from $(2,0)$ towards $(\frac{1}{2}, 2)$.

Solution $D_f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (e^y, xe^y)$

$$\vec{u} = (\tfrac{1}{2}, 2) - (2,0) = (-\tfrac{3}{2}, 2)$$

$$D_f \cdot \vec{u} = (e^y, xe^y) \cdot (-\tfrac{3}{2}, 2) = -\tfrac{3}{2}e^y + 2xe^y$$

$$D_f \cdot \vec{u}\Big|_{(x,y)=(2,0)} = -\tfrac{3}{2}e^y + 2xe^y\Big|_{(x,y)=(2,0)}$$

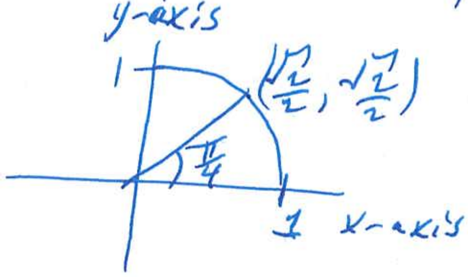
$$= -\tfrac{3}{2}e^0 + 2\cdot 2e^0 = -\tfrac{3}{2} + 4 = \tfrac{5}{2}.$$

$$\hat{u} = \frac{1}{\sqrt{\frac{9}{4}+4}}\vec{u} = \frac{1}{\sqrt{\frac{25}{4}}}\vec{u} = \frac{2}{5}(-\tfrac{3}{2}, 2) = (-\tfrac{3}{5}, \tfrac{4}{5}).$$

$$D_f \cdot \hat{u}\Big|_{(x,y)=(2,0)} = \frac{5}{2}\cdot\frac{2}{5} = 1.$$

Example 7.17 Find the directional derivative of $f(x,y) = \arcsin\left(\frac{x}{y}\right)$ at $(1,2)$ in the direction $\frac{\pi}{4}$ anticlockwise from the positive x-axis.

Solution



$$\hat{u} = \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}.$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y}\hat{i} + \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2}\hat{j}$$

So
$$\vec{\nabla} f \cdot \hat{u} = \frac{\sqrt{2}}{2y\sqrt{1-\left(\frac{x}{y}\right)^2}} + \frac{-\sqrt{2}\,x}{2y^2\sqrt{1-\left(\frac{x}{y}\right)^2}} \quad \text{and}$$

$$\vec{\nabla} f \cdot \hat{u}\Big|_{(x,y)=(1,2)} = \frac{\sqrt{2}}{2\cdot 2\sqrt{1-\left(\frac{1}{2}\right)^2}} - \frac{\sqrt{2}\cdot 1}{1\cdot 2^2\sqrt{1-\left(\frac{1}{2}\right)^2}}$$

$$= \frac{\sqrt{2}}{4}\left(\frac{1}{\sqrt{\frac{3}{4}}} - \frac{1}{2\sqrt{\frac{3}{4}}}\right) = \frac{\sqrt{2}}{4}\frac{2}{\sqrt{3}}\left(1-\frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{2} = \frac{1}{2\sqrt{2}\sqrt{3}} = \frac{1}{2\sqrt{6}}.$$