Example 1.31 Does $\displaystyle\sum_{n\in\mathbb{Z}_{\geq 0}} \frac{7}{2n^2+4n+3}$ converge or diverge?

Solution: $\displaystyle 0 < \frac{7}{2n^2+4n+3} < \frac{7}{2}\frac{1}{n^2}$ for $n\in\mathbb{Z}_{\geq 0}$.

So
$$\sum_{n\in\mathbb{Z}_{\geq 0}} 0 \leq \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{7}{2n^2+4n+3} \leq \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{7}{2}\frac{1}{n^2}.$$

Since

$\displaystyle\sum_{n\in\mathbb{Z}_{\geq 0}} 0 = 0$ converges and

$\displaystyle\frac{7}{2}\sum_{n\in\mathbb{Z}_{\geq 0}} \frac{1}{n^2} = \frac{7}{2}\left(\frac{\pi^2}{6}\right)$ converges then

$\displaystyle\sum_{n\in\mathbb{Z}_{\geq 0}} \frac{7}{2n^2+4n+3}$ converges.

Example 1.32 Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+1}$ converge or diverge?

Solution $\displaystyle\frac{n^2+4}{n^3+1} = \frac{\frac{1}{n}(n^3+1)+4-\frac{1}{n}}{n^3+1} = \frac{1}{n} + \frac{4-\frac{1}{n}}{n^3+1} > \frac{1}{n}$

for $n\in\mathbb{Z}_{>0}$. Since

$\displaystyle\sum_{n\in\mathbb{Z}_{>0}} \frac{1}{n}$ is infinite and does not converge

then $\displaystyle\sum_{n\in\mathbb{Z}_{>0}} \frac{n^2+4}{n^3+1}$ is infinite and does not converge.

Example 1.33 Does $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converge or diverge.

Solution: Let

$$L = \lim_{n\to\infty} \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \lim_{n\to\infty} \frac{10}{n+1} = 0.$$ Since $L<1$

the ratio test says that $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges.

More precisely,

$$\sum_{n=1}^{\infty} \frac{10^n}{n!} = \left[1 + 10 + \frac{1}{2!}10^2 + \frac{1}{3!}10^3 + \cdots\right] - 1 = e^{10} - 1.$$

Example 1.34 Does $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ converge or diverge?

Solution: Let

$$L = \lim_{n\to\infty} \frac{\frac{(2(n+1))!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)(n+1)}}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \left(\frac{2(n+1)-1}{n+1}\right)\left(\frac{2(n+1)}{n+1}\right) = \lim_{n\to\infty} \left(2 - \frac{1}{n+1}\right)2 = 2\cdot 2 = 4$$

Since $L>1$ the ratio test says that

$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ diverges.

Alternatively $\frac{(2n)!}{n!\,n!} = \binom{2n}{n} > 1$ and so

$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!} > \sum_{n=1}^{\infty} 1$ and so $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ is infinite and diverges