R.4. Exercises: Modules

Exercise R.4.1. — Let A be an abelian group with operation $+\colon A \times A \to A$. For each integer $j \in \mathbb{Z}$ and each $a \in A$ define

$$ja = \begin{cases} a + \cdots + a \ (j \text{ times}), & \text{if } j > 0, \\ 0, & \text{if } j = 0, \\ (-a) + (-a) + \cdots + (-a) \ (|j| \text{ times}), & \text{if } j < 0. \end{cases}$$

Define an action of the ring of integers $\mathbb{Z}$, with operations of addition and multiplication, by

$$\begin{array}{rccc} \alpha\colon & \mathbb{Z} \times A & \to & A \\ & (j, a) & \mapsto & ja \end{array}$$

Show that A with action of $\mathbb{Z}$ given by α is a left $\mathbb{Z}$-module.

Exercise R.4.2. —

(a) Let R be a ring. Define an action of R on itself by

$$\begin{array}{rccc} \alpha\colon & R \times R & \to & R \\ & (r, m) & \mapsto & rm. \end{array}$$

Show that R with the action α is a left R-module.

(b) Let I be an ideal of a ring R. Define an action of R on R/I by

$$\begin{array}{rccc} \beta\colon & R \times R/I & \to & R/I \\ & (r, s + I) & \mapsto & rs + I \end{array}$$

Show that R/I with the action β is a left R-module.

(c) Let I be an ideal of a ring R. Show that I is a submodule of the R-module R with action as given in (a).

(d) Let R be a ring. A **left ideal** of R is a subset $L \subseteq R$ such that
  (da) If $a, b \in L$ then $a + b \in L$.
  (db) If $l \in L$ and $r \in R$ then $rl \in L$.
Show that a subset $L \subseteq R$ is a left ideal of R if and only if L is a submodule of the left R-module R $\big($given by a)$\big)$.

(e) Give an example of a ring R and a left ideal L of R such that L is not an ideal of R.

Exercise R.4.3. — The **annihilator** of an element m of a left R-module M is the set

$$\mathrm{ann}_R(m) = \{r \in R \mid rm = 0\}.$$

The **annihilator** of a subset S of a left R-module M is the set

$$\mathrm{ann}_R(S) = \{r \in R \mid \text{if } s \in S \text{ then } rs = 0\}.$$

(a) Show that if S is a subset of an R-module M, then $\mathrm{ann}_R(S)$ is a left ideal of R.

(b) Give an example of a left R-module M and a subset $S \subseteq M$ such that $\mathrm{ann}_R(S)$ is not an ideal of R.

(c) Show that if M is a left R-module then $\mathrm{ann}_R(M)$ is an ideal of R.

(d) Show that if M is a left R-module and I is an ideal of R such that $I \subseteq \mathrm{ann}_R(M)$ then M, with action given by

$$\begin{array}{ccc} R/I \times M & \to & M \\ (r+I, m) & \mapsto & rm, \end{array}$$

is a left R/I-module.

Exercise R.4.4. —

(a) Show that the intersection of two submodules of a left R-module M is a submodule of M.
(b) Give an example to show that the union of two submodules of a left R-module M is not necessarily a submodule M.
(c) Let N and P be submodules of a left R-module M. Show that $N + P = \{n + p \mid n \in N \text{ and } p \in P\}$ is a submodule of M.
(d) Let N and P be submodules of a left R-module M. Show that $M \simeq N \oplus P$ if and only if $N \cap P = (0)$ and $M = N + P$.
(e) Let L be a left ideal of a ring R and let M be a left R-module. Show that $LM = \{lm \mid l \in L \text{ and } m \in M\}$ is a submodule of M.
(f) Let L_1 and L_2 be left ideals of R and define $L_1+L_2 = \{l_1+l_2 \mid l_1 \in L_1 \text{ and } l_2 \in L_2\}$. Show that if $L_1 \cap L_2 = (0)$ and $L_1 + L_2 = R$ then $M \simeq L_1M \oplus L_2M$.

Exercise R.4.5. — Let M be a left R-module and let S be a subset of M. Let $\mathcal{N}$ be the set of submodules N of M such that $S \subseteq N$. Define

$$N_S = \bigcap_{N \in \mathcal{N}} N.$$

(a) Show that N_S is a submodule of M.
(b) Show that $S \subseteq N_S$ since $S \subseteq N$ for every $N \in \mathcal{N}$.
(c) Show that if N is a submodule of M and $S \subseteq N$ then $N \supseteq N_S$.

Conclude that $N_S = (S)$. So (S) is the smallest submodule of M containing S.

Exercise R.4.6. — Let $f\colon M \to P$ be an R-module homomorphism.

(a) Let $N \subseteq M$ be a submodule of M and define

$$f(N) = \{f(n) \mid n \in N\}.$$

(aa) Show that $f(N) \subseteq \mathrm{im} f = f(M)$.
(ab) Show that $f(N)$ is a submodule of P.

$$\begin{array}{ccc} M & \xrightarrow{f} & P \\ & & \\ N & \longmapsto & f(N) \\ \cap\!| & & \cap\!| \\ M & \longmapsto & f(M) \quad = \mathrm{im} f. \end{array}$$

(b) Let Q be a submodule of P and define

$$f^{-1}(Q) = \{m \in M \mid f(m) \in Q\}.$$

(ba) Show that $f^{-1}(Q) \supseteq \ker f = f^{-1}\big((0)\big)$.
(bb) Show that $f^{-1}(Q)$ is a submodule of R.

$$\begin{array}{rccc} & M & \xrightarrow{f} & P \\ \\ & f^{-1}(Q) & \longmapsto & Q \\ & \cup\!\!| & & \cup\!\!| \\ \ker f = & f^{-1}\big((0)\big) & \longmapsto & (0). \end{array}$$

(c) (ca) Let N be a submodule of M and show that $N \subseteq f^{-1}\big(f(N)\big)$.

(cb) Give an example of a homomorphism $f\colon M \to P$ and a submodule N of M such that

$$N \neq f^{-1}\big(f(N)\big).$$

(cc) Show that if N is a submodule of M that contains $\ker f$ then $N = f^{-1}\big(f(N)\big)$.

(d) (da) Let Q be a submodule of P and show that $f\big(f^{-1}(Q)\big) \subseteq Q$.

(db) Give an example of a homomorphism $f\colon M \to P$ and a submodule Q such that $f\big(f^{-1}(Q)\big) \neq Q$.

(dc) Show that if Q is a submodule of P and $Q \subseteq \mathrm{im} f$ then $Q = f\big(f^{-1}(Q)\big)$.

(e) Conclude from (c) and (d) that there is a one-to-one correspondence between submodules of M that contain $\ker f$ and submodules of P that are contained in $\mathrm{im} f$.

$$\{\text{submdoules of } M \text{ containing } \ker f\} \quad \longleftrightarrow \quad \{\text{submodules of } P \text{ contained in } \mathrm{im}\, f\}$$

Exercise R.4.7. — Let N be a submodule of a left R-module M.

(a) The **inclusion** is the map $\iota\colon N \to M$ given by

$$\begin{array}{rccc} \iota\colon & N & \to & M \\ & n & \mapsto & n. \end{array}$$

Show that $\iota\colon N \to M$ is a well defined injective homomorphism.

(b) The **quotient map** is the map $\pi\colon M \to M/N$ given by

$$\begin{array}{rccc} \pi\colon & M & \to & M/N \\ & m & \mapsto & m+N. \end{array}$$

Show that $\pi\colon M \to M/N$ is a well defined surjective homomorphism and that $\mathrm{im}\pi = M/N$ and $\ker \pi = N$.

(c) Let P be a submodule of M. Show, using Ex. R.4.6, that

(ca) $P/N = \{p + N \mid p \in P\}$ is a submodule of M/N.

(cb) $P/N = \pi(P)$ and if P contains N then $\pi^{-1}\big(\pi(P)\big) = P$.

(cc) Conclude that there is a one-to-one correspondence

$$\{\text{submodules of } M \text{ containing } N\} \longleftrightarrow \{\text{submodules of } M/N\}.$$

Exercise R.4.8. — An **exact sequence**

$$\cdots \to M_{i-1} \overset{f_{i-1}}{\to} M_i \overset{f_i}{\to} M_{i+i} \to \cdots$$

is a sequence of left R-modules and R-module homomorphisms $f_i\colon M_i \to M_{i+1}$, such that

$$\ker f_i = \mathrm{im} f_{i-1}$$

A **short exact sequence** is an exact sequence of the form

$$(0) \to N \overset{g}{\to} M \overset{f}{\to} P \to (0).$$

(a) Show that in the above short exact sequence g is always injective and f is always surjective.

(b) Let $f\colon M \to P$ be a homomorphism and show that the sequence

$$(0) \to \ker f \to M \to \mathrm{im} f \to (0)$$

is always exact.

(c) Let N be a submodule of a left R-molule M. Let $\iota\colon N \to M$ be the canonical injection and let $\pi\colon M \to M/N$ be the canonical surjection. Show that

$$(0) \to N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \to (0)$$

is a short exact sequence.

Exercise R.4.9. — Let N be a submodule of M and let P be a submodule of M containing N. Then, by Ex.R.4.7(ca), P/N is a submodule of M/N.
Let $\frac{M/N}{P/N}$ be the quotient module and let

$$\pi_2\colon M/N \to \frac{M/N}{P/N} \qquad \text{be the canonical surjection.}$$

Let $\pi_1\colon M \to M/N$ be the canonical surjection so that

$$(\pi_1 \circ \pi_2)\colon M \xrightarrow{\pi_1} M/N \xrightarrow{\pi_2} \frac{M/N}{P/N}.$$

(a) Show that $\mathrm{im}(\pi_1 \circ \pi_2) = \dfrac{M/N}{P/N}$.

(b) Show that $\ker(\pi_1 \circ \pi_2) = P$.

(c) Using Theorem R.2.6, conclude that $M/P \simeq \dfrac{M/N}{P/N}$ as left R-modules.

Exercise R.4.10. — Let N be a submodule of M and let P be any submodule of M. Let

$$\begin{array}{rccc} \pi\colon & P & \to & M/N \\ & p & \mapsto & p+N \end{array}$$

be the restriction of the canonical surjection $\pi\colon M \to M/N$ to P.

(a) Show that $\ker \pi = P \cap N$.

(b) Show that $\mathrm{im}\pi = \dfrac{P+N}{N} = \{p+N \mid p \in P\}$.

(c) Using Theorem R.2.6(c), conclude that $\dfrac{P}{P\cap N} \simeq \dfrac{P+N}{N}$.

(d) Let L be a left ideal of a ring R and let N be a submodule of left R-module M. Let

$$LM = \{lm \mid l \in L \text{ and } m \in M\} \quad \text{and let}$$
$$L(M/N) = \{l(m+N) \mid l \in L \text{ and } m+N \in M/N\}.$$

Show that

$$L\left(\frac{M}{N}\right) \simeq \frac{LM+N}{N} \qquad \text{as } R\text{-modules.}$$

Exercise R.4.11. — Let N_1 be a submodule of a left R-module M_1 and let N_2 be a submodule of a left R-module M_2.

(a) Show that $N_1 \oplus N_2$ is a submodule of the left R-module $M_1 \oplus M_2$.

(b) Let $\pi_1\colon M_1 \to M_1/N_1$ and $\pi_2\colon M_2 \to M_2/N_2$ be the canonical projections. Define a map

$$\begin{array}{rccc} (\pi_1 \oplus \pi_2)\colon & M_1 \oplus M_2 & \to & M_1/N_1 \oplus M_2/N_2 \\ & (m_1, m_2) & \mapsto & (m_1 + N_1, m_2 + N_2). \end{array}$$

Show that $\pi_1 \oplus \pi_2$ is a well defined surjective R-module homomorphism.

(c) Show that $\ker(\pi_1 \oplus \pi_2) = N_1 \oplus N_2$.

(d) Using Theorem R.2.6, conclude that

$$\frac{M_1 \oplus M_2}{N_1 \oplus N_2} \simeq \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}.$$

Exercise R.4.12. — A **right R-module** is a set M with an operation addition $+\colon M \times M \to M$ and a **right action** $\times\colon M \times R \to M$ (we write $m_1 + m_2$ instead of $+(m_1, m_2)$ and mr instead of $\times(m, r)$) such that

(a) If $m_1, m_2, m_3 \in M$ then $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$.
(b) If $m_1, m_2 \in M$ then $m_1 + m_2 = m_2 + m_1$.
(c) There exists a **zero**, $0 \in M$, such that $0 + m = m$ for all $m \in M$.
(d) If $m \in M$ then there exists an **additive inverse**, $-m \in M$, such that $m + (-m) = 0$.
(e) If $r_1, r_2 \in R$ and $m \in M$ then $m(r_1 r_2) = (m r_1) r_2$.
(f) If $m \in M$ then $m \cdot 1 = m$
(g) If $r \in R$ and $m_1, m_2 \in M$ then $(m_1 + m_2)r = m_1 r + m_2 r$.
(h) If $r_1, r_2 \in R$ and $m \in M$ then $m(r_1 + r_2) = m r_1 + m r_2$.

(a) Copy all of §R.2 over, change left to right everywhere, and make the appropriate changes in the definitions and the proofs. In this way, prove analogues of all of the results in §R.2 on left R-modules for right R-modules.

(b) Let R be a commutative ring and let M be a left R-module. Define a right action of R on M by

$$\begin{array}{rccc} \beta\colon & M \times R & \to & M \\ & (m, r) & \mapsto & rm. \end{array}$$

Show that M is a right R-module with right action given by β. This shows that if R is commutative then

every left R-module is also a right R-module.

Warning: This is true only if R is commutative. In general, something true for the left R-module over a *particular* ring R is not necessarily true for the right R-modules over that ring.