F.2. Vector spaces

Definition F.2.1. — Let $\mathbb{F}$ be a field with identity $1 \in \mathbb{F}$.

- An **$\mathbb{F}$-vector space** is a set V with functions $+: V \times V \to V$ (**addition**) and (**scalar multiplication**) $\times: \mathbb{F} \times V \to V$ $\big($we write $v + w$ instead of $+(v, w)$ and cv instead of $\times(c, v)\big)$ such that
  (a) If $v_1, v_2, v_3 \in V$ then $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$,
  (b) If $v_1, v_2 \in V$ then $v_1 + v_2 = v_2 + v_1$,
  (c) There exists a **zero**, $0 \in V$, such that if $v \in V$ then $0 + v = v$,
  (d) If $v \in V$ there exists an **additive inverse**, $-v \in V$, such that $v + (-v) = 0$.
  (e) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $c_1(c_2 v) = (c_1 c_2)v$.
  (f) If $v \in V$ then $1 \cdot v = v$.
  (g) If $c \in \mathbb{F}$ and $v_1, v_2 \in V$ then $c(v_1 + v_2) = cv_1 + cv_2$.
  (h) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $(c_1 + c_2)v = c_1 v + c_2 v$.
- A **subspace** W of an $\mathbb{F}$-vector space V is a subset $W \subseteq V$ such that
  (a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
  (b) $0 \in W$,
  (c) If $w \in W$ then $-w \in W$,
  (d) If $w \in W$ and $c \in \mathbb{F}$ then $cw \in W$.
- The **zero space** $\{0\}$ is the set containing only 0 with operations $0 + 0 = 0$ and $c \cdot 0 = 0$ for $c \in \mathbb{F}$.

A vector space is just a module over a field. Properties (a), (b), (c), and (d) in the definition of an $\mathbb{F}$-vector space imply that a vector space is an abelian group with an action of the field $\mathbb{F}$.

HW: Show that the element $0 \in V$ is unique.

HW: Show that if $v \in V$ then the element $-v \in V$ is unique.

HW: Show that if V is an $\mathbb{F}$-vector space and $v \in V$ then $0 \cdot v = 0$. (The 0 on the left hand side of this equation is an element of $\mathbb{F}$ and the 0 on the right hand side is an element of V.)

HW: Show that if $c \in \mathbb{F}$ then $c0 = 0$. (The 0 on both sides of this equation is the zero in V.)

HW: Let V be an $\mathbb{F}$-vector space and let $c \in \mathbb{F}$ and $v \in V$. Show that $c \cdot v = 0$ if and only if either $c = 0$ or $v = 0$.

Important examples of vector spaces are:
  (a) $\mathbb{R}^k$ (is an $\mathbb{R}$-vector space) and $\mathbb{C}^k$ (is a $\mathbb{C}$-vector space).
  (b) If $\mathbb{F}$ is a field then $\mathbb{F}^k$ is an $\mathbb{F}$-vector space.

Linear transformations are used to compare vector spaces. A linear transformation must preserve the structures that distinguish an $\mathbb{F}$-vector space: the addition and the scalar multiplication.

Definition F.2.2. —

- A **linear transformation** is a function $T: V \to W$ between $\mathbb{F}$-vector spaces V and W such that

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(a) If $v_1, v_2 \in V$ then $T(v_1 + v_2) = T(v_1) + T(v_2)$,
(b) If $c \in \mathbb{F}$ and $v \in V$ then $T(cv) = cT(v)$.

- A **vector space isomorphism** is a linear transformation $T\colon V \to W$ such that the inverse function $T^{-1}\colon W \to V$ exists and is a linear transformation.
- Two $\mathbb{F}$-vector spaces V and W are **isomorphic**, $V \simeq W$, if there exists a vector space isomorphism $T\colon V \to W$ between them.

Two $\mathbb{F}$-vector spaces are isomorphic if the elements of the vector spaces and the operations and the actions match up exactly. Think of two vector spaces that are isomorphic as being "the same".

HW: Let $T\colon V \to W$ be a linear transformation. Show that T is a vector space isomorphism if and only if T is bijective.

Proposition F.2.1. — *Let $T\colon V \to W$ be a linear transformation. Let 0_V and 0_W be the zeros for V and W respectively. Then*
(a) $T(0_V) = 0_W$, *and*
(b) *If $v \in V$ then $T(-v) = -T(v)$.*

F.2.1. Cosets. —

Definition F.2.3. —

- A **subgroup** of an $\mathbb{F}$-vector space V is a subset $W \subseteq V$ such that
  (a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$.
  (b) $0 \in W$.
  (c) If $w \in W$ then $-w \in W$.

Let V be an $\mathbb{F}$-vector space and let W be a subgroup of V. We will use the subgroup W to divide up the module V.

Definition F.2.4. — Let V be an $\mathbb{F}$-vector space and let W be a subgroup of V.

- A **coset** of W in V is a set $v + W = \{v + w \mid w \in W\}$ where $v \in V$.
- V/W (pronounced "V **mod** W") is the set of cosets of W in V.

Proposition F.2.2. — *Let V be an $\mathbb{F}$-vector space and let W be a subgroup of V. Then the cosets of W in V partition V.*

Notice the analogy between Proposition F.2.2 and Proposition R.1.2 and Proposition R.2.2 and Proposition G.1.2.

F.2.2. Quotient Spaces $\leftrightarrow$ Subspaces. — Let V be an $\mathbb{F}$-vector space and let W be a subspace of V. We can try to make the set V/W of cosets of W in V into an $\mathbb{F}$-vector space by defining an addition operation and an action of $\mathbb{F}$.

Theorem F.2.3. — *Let W be a subgroup of a vector space V over a field $\mathbb{F}$. Then W is a subspace of V if and only if V/W with operations given by*

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \quad \textit{and} \quad c(v + W) = cv + W,$$

is an $\mathbb{F}$-vector space.

Notice the analogy between Theorem F.2.3, Theorem R.2.3, Theorem R.1.3 and Theorem G.1.5.

Definition F.2.5. —

- The **quotient space** V/W is the $\mathbb{F}$-vector space of cosets of a subspace W of an $\mathbb{F}$-vector space V with operations given by

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \text{and} \qquad (v + W) = cv + W.$$

We have made V/W into a vector space when W is a subspace of V.

HW: Show that if $W = V$ then $V/W \simeq \{0\}$.

F.2.3. Kernel and image of a linear transformation. —

Definition F.2.6. — Let $T\colon V \to W$ be a linear transformation.

- The **kernel**, or **null space**, of T is the set
$$\ker T = \{v \in V \mid T(v) = 0_W\},$$
where 0_W is the zero element of W.
- The **image** of T is the set
$$\operatorname{im} T = \{T(v) \mid v \in V\}.$$

Proposition F.2.4. — *Let $T\colon V \to W$ be a linear transformation. Then*

(a) $\ker T$ *is a subspace of* V.

(b) $\operatorname{im} T$ *is a subspace of* W.

Proposition F.2.5. — *Let $T\colon V \to W$ be a linear transformation. Let 0_V be the zero in V. Then*

(a) $\ker T = \{0_V\}$ *if and only if T is injective.*

(b) $\operatorname{im} T = W$ *if and only if T is surjective.*

Notice that the proof of Proposition F.2.5 (b) does not use the fact that $T\colon V \to W$ is a linear transformation, only the fact that $T\colon V \to W$ is a function.

Theorem F.2.6. —

(a) Let $T\colon V \to W$ be a linear transformation and let $N = \ker T$. Define
$$\begin{array}{rccc} \hat{T}\colon & V/\ker T & \to & W \\ & v + N & \mapsto & f(v). \end{array}$$
Then $\hat{T}$ is a well defined injective linear transformation.

(b) Let $T\colon V \to W$ be a linear transformation and define
$$\begin{array}{rccc} T'\colon & V & \to & \operatorname{im} T \\ & v & \mapsto & T(v). \end{array}$$
Then T' is a well defined surjective linear transformation.

(c) If $T\colon V \to W$ is a linear transformation, then
$$V/\ker T \simeq \operatorname{im} T,$$
where the isomorphism is a vector space isomorphism.

F.2.4. Direct Sums. — Suppose V and W are $\mathbb{F}$-vector spaces. The idea is to make the set $V \times W$ into a vector space.

Definition F.2.7. —

- The **direct sum** of $V \oplus W$ of two vector spaces V and W over a field $\mathbb{F}$ is the set $V \times W$ with operations given by
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v, w) = (cv, cw),$$
for $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$ and $c \in \mathbb{F}$. The operations in $V \oplus W$ are **componentwise**.
- More generally, given vector spaces $V_1, V_2, \ldots, V_n$ over F the **direct sum** $V_1 \oplus \cdots \oplus V_n$ is the set $V_1 \times \cdots \times V_n$ with the operations given by
$$(v_1, \ldots, v_i, \ldots, v_n) + (w_1, \ldots, w_i, \ldots, w_n) = (v_1 + w_1, \ldots, v_i + w_i, \ldots, v_n + w_n), \quad \text{and}$$
$$c(v_1, \ldots, v_i, \ldots, v_n) = (cv_1, \ldots, cv_i, \ldots, cv_n),$$
where $v_i, w_i \in V_i$, $c \in \mathbb{F}$, and $v_i + w_i$ and cv_i are given by the operations in V_i.

HW: Show that these are good definitions, i.e., that as defined above, $V \oplus W$ and $V_1 \oplus \cdots \oplus V_n$ are vector spaces over $\mathbb{F}$ with zeros given by $(0_V, 0_W)$ and $(0_{V_1}, \ldots, 0_{V_n})$ respectively. (0_{V_i} denotes the zero element in V_i.)

F.2.5. Further Definitions. —

Definition F.2.8. — Let V be an $\mathbb{F}$- vector space and let S be a subset of V.

- The **span of** S or the **subspace generated by** S, is the subspace $\mathrm{span}_{\mathbb{F}}(S)$ of V such that
  (a) $S \subseteq \mathrm{span}_{\mathbb{F}}(S)$,
  (b) If W is a subspace of V and $S \subseteq W$ then $\mathrm{span}_{\mathbb{F}}(S) \subseteq W$.

The subspace $\mathrm{span}_{\mathbb{F}}(S)$ is the smallest subspace of V containing S. Think of $\mathrm{span}_{\mathbb{F}}(S)$ as gotten by adding to S exactly those elements of V that are needed to make a subspace.

Definition F.2.9. — Let V be an $\mathbb{F}$- vector space and let S be a subset of V.

- The **span of** S is the subspace of V
$$\mathrm{span}_{\mathbb{F}}(S) = \{c_1 v_1 + \ldots + c_k v_k \mid k \in \mathbb{Z}_{>0},\ v_1, \ldots, v_k \in V, c_1, \ldots, c_k \in \mathbb{F}\}$$
- The set S is **linearly independent** f it satisfies:
  if $k \in \mathbb{Z}_{>0}$ and $v_1, \ldots, v_k \in S$ and $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1 v_1 + \cdots + c_k v_k = 0$
  then $c_1 = 0$, $c_2 = 0, \ldots, c_k = 0$.
- A **basis** of V is a subset $B \subseteq V$ such that
  (a) $\mathrm{span}_{\mathbb{F}}(B) = V$,
  (b) B is linearly independent.
- The **dimension** of V is $\dim(V) = \mathrm{Card}(B)$, where B is a basis of V.

Proposition F.2.7. — *Let V be an $\mathbb{F}$-vector space and let B be a subset of V. The following are equivalent:*

(a) B is a basis of V.
(b) B is a minimal element of $\{S \subseteq V \mid span_{\mathbb{F}}(S) = V\}$.
(c) B is a maximal element of $\{L \subseteq V \mid L \text{ is linearly independent}\}$.

(In (b) and (c) the ordering is by inclusion.)

Theorem F.2.8. — *Let V be an $\mathbb{F}$-vector space. Then*

(a) V has a basis, and

(b) Any two bases of V have the same number of elements.