GTLA Lecture 11.08.2020

Osmosis Topics

- Sets
- Functions
- Relations
- Orders
- The integers
- Cardinalities
- Equivalence relations

The binomial theorem

Let $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, 1, \ldots, n\}$.

Define
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Theorem Assume $xy = yx$.

(a) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ if $k \in \{1, \ldots, n-1\}$

$\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

(b) $(x+y)^n$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

(c) Let S be a set of cardinality n.

$\binom{n}{k} = \left(\begin{array}{l} \#\text{ of subsets of } S \\ \text{cardinality } k \end{array} \right)$

(d) $e^{x+y} = e^x e^y$

<u>Sketch of (c)</u> A term $x^{n-k}y^k$ in

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}}$$

comes from choosing k factors to contribute y and the others contribute x.

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= xxx + xxy + xyx + yxx$$
$$+ xyy + yxy + yyx$$
$$+ yyy$$

<u>Corollary</u> $\binom{n}{k} \in \mathbb{Z}$. $\quad \dfrac{n!}{k!(n-k)!} \in \mathbb{Z}_{\geq 0}$

$$e^{x+y} = 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \cdots$$

$$\begin{aligned}
= \quad & 1 \\
& + x + y \\
& \frac{1}{2!}(x^2 + 2xy + y^2) \\
& + \frac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^3) \\
& \vdots
\end{aligned}$$

$$\begin{aligned}
= \quad & 1 \\
& + x + y \\
& + \frac{1}{2!}x^2 + xy + \frac{1}{2!}y^2 \\
& + \frac{1}{3!}x^3 + \frac{1}{2!}x^2y + \frac{1}{2!}xy^2 + \frac{1}{3!}y^3 \\
& \vdots
\end{aligned}$$

$$= e^x + e^x y + e^x \frac{1}{2!}y^2 + e^x \frac{1}{3!}y^3 + \cdots$$

$$= e^x\left(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots\right)$$

$$= e^x e^y$$

So $\boxed{e^{x+y} = e^x e^y}$

The integers

Existence of $\gcd(a,b)$

<u>Proposition</u> Let $a,b \in \mathbb{Z}_{>0}$. Then there exists $\ell \in \mathbb{Z}_{>0}$ such that

$$\ell\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$$

<u>Proof</u> Assume $a,b \in \mathbb{Z}_{>0}$.

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that $\ell\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$.

Let $\ell = \min(\mathbb{Z}_{>0} \cap (a\mathbb{Z}+b\mathbb{Z}))$.

To show: $\ell\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$

To show: (a) $\mathbb{Z}_{>0} \cap (a\mathbb{Z}+b\mathbb{Z}) \neq \emptyset$

(b) If $S \subseteq \mathbb{Z}_{\geq 0}$ and $S \neq \emptyset$ then there exists a unique minimal element in S.

(a) Since $a \in \mathbb{Z}_{>0}$ and

$$a = a \cdot 1 + \underbrace{b \cdot 0} \in a\mathbb{Z} + b\mathbb{Z}$$

then $a \in \mathbb{Z}_{\geq 0} \cap (a\mathbb{Z}+b\mathbb{Z})$.

So $\mathbb{Z}_{\geq 0} \cap (a\mathbb{Z}+b\mathbb{Z}) \neq \emptyset$.

(b) Assume $S \subseteq \underbrace{\mathbb{Z}_{\geq 0}}$, and $S \neq \emptyset$.

To show: There exist a unique $\underbrace{\text{minimal}}$ element in S.

Since $S \neq \emptyset$ then there exists $a \in S$.

If $0 \in S$ then $0 = \min(S)$

If $0 \notin S$ and $1 \in S$ then $1 = \min(S)$

If $0, 1 \notin S$ and $2 = 1+1 \in S$ then $2 = \min(S)$

If $0, 1, 2 \notin S$ and $3 \in S$ then $3 = \min(S)$

$\vdots$

This has to stop before the a^{th} step since $a \in S$ and $a = 1+1+\cdots+1$.

A *poset*, or *partially ordered set*, is a set S with a relation $\leq$ such that

(a) If $x \in S$ then $x \leq x$

(b) If $x, y, z \in S$ and $x \leq y$ and $y \leq z$ then $x \leq z$

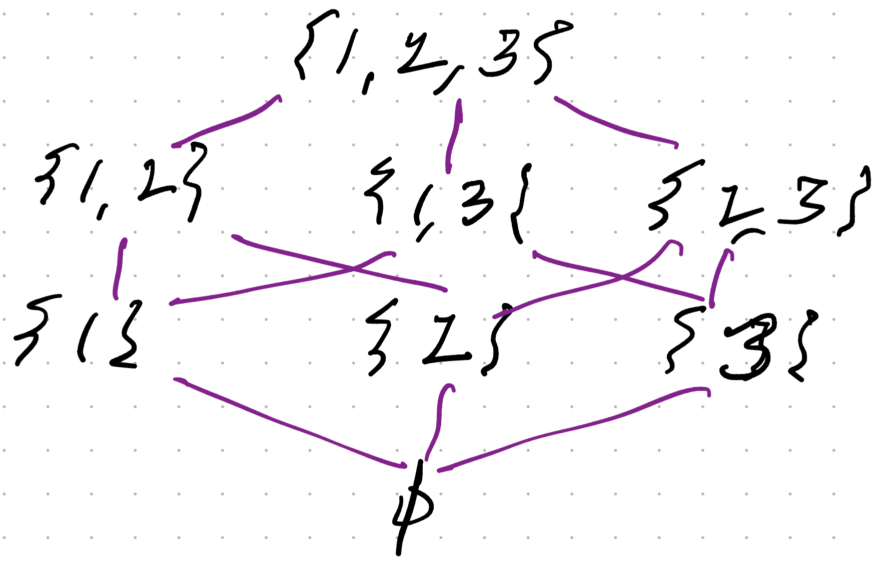
(c) If $x, y \in S$ and $x \leq y$ and $y \leq x$ then $x = y$.

A *total order* also satisfies

(d) If $x, y \in S$ then $x \leq y$ or $y \leq x$.

Example

$S = \{\text{subsets of } \{1,2,3\}\}$ ordered by inclusion.



If $x = \{1\}$ and $y = \{3\}$
then $x \not\leq y$ and $y \not\leq x$

The integers $\mathbb{Z}$.

Ordered commutative ring.

1. $\mathbb{Z}$ is a set.
2. $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
   $(a, b) \mapsto a + b$ addition
3. $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
   $(a, b) \mapsto ab$ multiplication
4. $x \leq y$ if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $x + n = y$.

which satisfy
(Fa), ..., (Fi), (Pa), ..., (Pd), (OFa) and (OFb)

$\mathbb{Z}_{>0} = \{1, 1+1, 1+1+1, \dots\}$

$= \{1, 2, 3, \dots\}$ <u>add</u>

$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$

with $0 + x = x$ and $x + 0 = x$.

$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$

<u>with</u>

$-x + x = 0$

$\mathbb{Z}_{>0}$ is the free monoid without identity <u>generated by</u> 1.

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<u>Prop</u> $\mathbb{Z}$, addition

<u>Let $k \in \mathbb{Z}$</u>. Then there exists a unique function $f : \mathbb{Z} \to \mathbb{Z}$ such that

(a) If $x, y \in \mathbb{Z}$ then

$$f(x+y) = f(x) + f(y)$$

(b) $f(1) = k$.

One of a, b, or c satisfies

x is divisible by 3

or

x is divisible by 4

or

x is divisible by 5.

$$\mathbb{K}[x] = \left\{ a_0 + a_1 x + \cdots + a_\ell x^\ell \,\middle|\, \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ \text{and} \\ a_0, \ldots, a_\ell \in \mathbb{K} \end{array} \right\}$$

Polynomial with coefficients in $\mathbb{K}$