Eigenvectors and nullspaces 18.08.2020 (1)

GTLA lecture

Let $A \in M_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$ and $\phi \in \mathbb{F}^n$ with $\phi \neq 0$.

Show that ϕ is an eigenvector of eigenvalue λ if and only if $\phi \in \ker(\lambda - A)$.

Proof $\Rightarrow$ Assume ϕ is an eigenvector of eigenvalue λ.

So $A\phi = \lambda\phi$.

So $\lambda\phi - A\phi = 0$. So $(\lambda - A)\phi = 0$.

So $\phi \in \ker(\lambda - A)$.

$\Leftarrow$ Assume $\phi \in \ker(\lambda - A)$

Then $(\lambda - A)\phi = 0$. So $\lambda\phi = A\phi$.

So ϕ is an eigenvector of eigenvalue λ. //.

<u>Eigenvectors and A-invariant subspaces</u>

Let $A \in M_n(\mathbb{F})$ and $p \in \mathbb{F}^n$ with $p \neq 0$. Show that p is an eigenvector of A if and only if $\mathbb{F}p$ is A-invariant.

<u>Proof</u> $\Rightarrow$ Assume p is an eigenvector of A.
Then there exists $\lambda \in \mathbb{F}$ such that $Ap = \lambda p$.
To show: If $u \in \mathbb{F}p$ then $Au \in \mathbb{F}p$
Assume $u \in \mathbb{F}p$.
Then there exists $c \in \mathbb{F}$ such that $u = cp$.
To show: $Au \in \mathbb{F}p$

$$Au = Acp = cAp = c\lambda p = (c\lambda)p \in \mathbb{F}p.$$

So $\mathbb{F}p$ is A-invariant.

$\Leftarrow$ Assume $\mathbb{F}p$ is A-invariant
To show: p is an eigenvector of A.
Since $\mathbb{F}p$ is A-invariant and $p \in \mathbb{F}p$
then $Ap \in \mathbb{F}p$.
So there exists $\lambda \in \mathbb{F}$ such that $Ap = \lambda p$.
So p is an eigenvector of A. //

We know $Ap \in \mathbb{F}p$.

So there exists $\lambda \in \mathbb{F}$ such that $Ap = \lambda p$.

So p is an eigenvector of A. //.

<u>Proposition</u> Let $A \in M_n(\mathbb{F})$.

If $p_1, \ldots, p_k$ are eigenvectors of A
of eigenvalues $\lambda_1, \ldots, \lambda_k$
and $\lambda_1, \ldots, \lambda_k$ are all distinct
then $p_1, \ldots, p_k$ are linearly independent.

<u>Proof</u> Assume $p_1, \ldots, p_k$ are eigenvectors of A
of eigenvalues $\lambda_1, \ldots, \lambda_k$ and $\lambda_1, \ldots, \lambda_k$
are all distinct.

To show: $p_1, \ldots, p_k$ are linearly independent.

To show: If $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1 p_1 + \cdots + c_k p_k = 0$
then $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

Assume $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1 p_1 + \cdots + c_k p_k = 0$.

To show: $c_1 = 0, \ldots, c_k = 0$.

To show: If $j \in \{1, \ldots, k\}$ then $c_j = 0$.

Assume $j \in \{1, \ldots, k\}$.

To show $c_j = 0$.

$$0 = (A - \lambda_1) \cdots (A - \lambda_{j-1})(A - \lambda_{j+1}) \cdots (A - \lambda_k)(c_1 p_1 + \cdots + c_k p_k)$$

$= c_1(\lambda_1-\lambda_1)(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_{j-1})(\lambda_1-\lambda_{j+1})\cdots(\lambda_1-\lambda_k)p_1$

$+ c_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_2)\cdots(\lambda_2-\lambda_{j-1})(\lambda_2-\lambda_{j+1})\cdots(\lambda_2-\lambda_k)p_2$

$+ \cdots$

$+ c_k(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)\cdots(\lambda_k-\lambda_{j-1})(\lambda_k-\lambda_{j+1})\cdots(\lambda_k-\lambda_k)p_k$

$= 0 + 0 + \cdots + c_j(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\cdots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})$

$\cdots(\lambda_j-\lambda_k)p_j$

$+ 0 + \cdots + 0 = c_j\mu_j p_j$.

Since the $\lambda_1, \dots, \lambda_k$ are all distinct

$\mu_j = (\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\cdots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\cdots(\lambda_j-\lambda_k) \neq 0$.

Since $p_j \neq 0$ then $c_j = 0$.

So $p_1, \dots, p_k$ are linearly independent. //

<u>Proposition</u> If A is diagonalisable with $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ then $\det(A) = \lambda_1 \cdots \lambda_n$.

<u>Proof</u> Assume $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

To show: $\det(A) = \lambda_1 \cdots \lambda_n$.

$\det(A) = \det(A)\det(P)^{-1}\det(P)$

$= \det(P)^{-1}\det(A)\det(P) = \det(P^{-1})\det(A)\det(P)$

$= \det(P^{-1}AP) = \det\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \lambda_1 \cdots \lambda_n$ //.