Let $A \in M_n(\mathbb{F})$.

The matrix A is <u>diagonalizable</u> if there exists $P \in GL_n(\mathbb{F})$ and $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

<u>Determinant of a diagonalizable matrix</u>

Assume $P^{-1}AP = D$ with $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Then

$$\begin{aligned} \det(A) &= \det(P)^{-1}\det(P)\det(A) \\ &= \det(P)^{-1}\det(A)\det(P) \\ &= \det(P^{-1})\det(A)\det(P) \\ &= \det(P^{-1}AP) = \det(D) \\ &= \lambda_1 \cdots \lambda_n . \end{aligned}$$

<u>Characteristic polynomial of a diagonalizable matrix</u>

$$\begin{aligned} \det(x-A) &= \det(P)^{-1}\det(P)\det(x-A) \\ &= \det(P)^{-1}\det(x-A)\det(P) \\ &= \det(P^{-1})\det(x-A)\det(P) \\ &= \det(P^{-1}(x-A)P) \\ &= \det(P^{-1}xP - P^{-1}AP) \\ &= \det(xP^{-1}P - D) = \det(x-D) \end{aligned}$$

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$$= \det\left(\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} - \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} x-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & x-\lambda_n \end{pmatrix}\right) =$$

$$= (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n).$$

<u>Theorem</u> Let $A \in M_n(\mathbb{F})$. Then A is diagonalisable if and only if there exist n-linearly independent eigenvectors of A.

<u>Sketch of proof</u> Assume $P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Then $AP = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Let $p_1 = \begin{pmatrix} p_{11} \\ \vdots \\ p_{n1} \end{pmatrix}, p_2 = \begin{pmatrix} p_{12} \\ \vdots \\ p_{n2} \end{pmatrix}, \ldots, p_n = \begin{pmatrix} p_{1n} \\ \vdots \\ p_{nn} \end{pmatrix}$

be the columns of A.

Since $AP = P\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$= \begin{pmatrix} p_{11}\lambda_1 & \cdots & p_{1n}\lambda_n \\ \vdots & & \vdots \\ p_{n1}\lambda_1 & \cdots & p_{nn}\lambda_n \end{pmatrix} \quad \text{then}$$

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$$Ap_j = A\begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} = \begin{pmatrix} p_{1j}\lambda_j \\ \vdots \\ p_{nj}\lambda_j \end{pmatrix} = \lambda_j \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} = \lambda_j p_j .$$

So $p_1, \dots, p_n$ are eigenvectors of A (with eigenvalues $\lambda_1, \dots, \lambda_n$).

Since P is invertible then the columns of P are linearly independent.

So $p_1, \dots, p_n$ are linearly independent. //

<u>Not diagonalizable over $\mathbb{R}$, but which is diagonalizable over $\mathbb{C}$</u>

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\det(x - A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \det\begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$$

$$= x^2 - (-1) = x^2 + 1 .$$

There does not exist $\lambda \in \mathbb{R}$ such that $\lambda^2 + 1 = 0$.

So $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has no eigenvectors over $\mathbb{R}$.

Let $p_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $p_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Then

$$Ap_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} = i\begin{pmatrix} 1 \\ i \end{pmatrix} = i p_1$$

$$Ap_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ -i \end{pmatrix} = -i p_2$$

So, letting $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ then

$$P^{-1}AP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and A is diagonalizable over $\mathbb{C}$.

<u>Theorem</u> If $\mathbb{F}$ is algebraically closed and $A \in M_n(\mathbb{F})$ then A has an eigenvector (over $\mathbb{F}$).

<u>Proof</u> Assume $\mathbb{F}$ is algebraically closed and $A \in M_n(\mathbb{F})$.

Since $\mathbb{F}$ is algebraically closed then the characteristic polynomial $\det(x - A)$ has a root.

So there exists $\lambda \in \mathbb{F}$ such that $\det(\lambda - A) = 0$.

Since $\det(\lambda - A) = 0$ then $\ker(\lambda - A) \neq 0$.

So there exists $p \in \ker(\lambda - A)$ with $p \neq 0$.

Then p is an eigenvector of A, since $(\lambda - A)p = 0$ gives $Ap = \lambda p$. //

2x2 matrix in $M_2(\mathbb{C})$ with only one eigenvector

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C})$.

Then $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

So $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of eigenvalue 1.

$$\det(x - A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$
$$= \det\begin{pmatrix} x-1 & -1 \\ 0 & x-1 \end{pmatrix} = (x-1)^2.$$

So the other eigenvector, if it exists has eigenvalue 1.

If $p = \begin{pmatrix} 1 \\ c \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ c \end{pmatrix}$

then $c+1 = 1$ and $c = 0$.

If $p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one linearly independent eigenvector.

<u>Note:</u> $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has 2 linearly independent eigenvectors (for example $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$).