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GTLA Lecture (1)

The *minimal polynomial of A* is the smallest degree monic polynomial $m_A(x)$ such that $m_A(A)=0$.

The *characteristic polynomial of A* is

$$\det(x-A)$$

Theorem (Cayley-Hamilton).

$\det(x-A)$ is a multiple of $m_A(x)$.

Let $d\in\mathbb{Z}_{>0}$ and $\lambda\in\mathbb{C}$. The *Jordan block* of size d and eigenvalue λ is

$$J_d(\lambda)=\begin{pmatrix}\lambda & 1 & & 0\\ & \lambda & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{pmatrix} \text{ in } M_d(\mathbb{C})$$

Then $\det(x-J_d(\lambda))=(x-\lambda)^d$ and

$$m_{J_d(\lambda)}(x)=(x-\lambda)^d.$$

If $d_1,\ldots,d_k\in\mathbb{Z}_{>0}$ and $A=J_{d_1}(\lambda)\oplus\cdots\oplus J_{d_k}(\lambda)$ then

$\det(x-A)=(x-\lambda)^{d_1+d_2+\cdots+d_k}$ and

$$m_A(x)=(x-\lambda)^{\max(d_1,\ldots,d_k)}.$$

__Proposition__ Let $A \in M_n(\mathbb{F})$ and $P \in GL_n(\mathbb{F})$.

(a) $\det(P^{-1}AP) = \det(A)$.

(b) $\det(x - P^{-1}AP) = \det(x - A)$

(c) $m_{P^{-1}AP}(x) = m_A(x)$

(d) $\ker(P^{-1}AP) = P^{-1}\ker(A)$.

Find. $\ker(PAQ)$:

(a) Let $v \in \ker(AQ)$. Then $AQv = 0$.

So $Qv \in \ker(A)$

So $v \in Q^{-1}\ker(A)$

So $\ker(AQ) = Q^{-1}\ker(A)$

(b) Let $v \in \ker(PA)$. Then $PAv = 0$.

So $Av = 0$. So $v \in \ker(A)$.

So $\underline{\ker(PA) = \ker(A)}$.

So

$$\boxed{\ker(PAQ) = Q^{-1}\ker(A).}$$

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Examples

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad m_{A_1}(x) = x-1$

$\det_{A_1}(x - A_1) = (x-1)^2$

2 linearly indep. eigenvectors over $\mathbb{C}$

$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad m_{A_2}(x) = (x-1)^2$

$\det(x - A_2) = (x-1)^2$

1 linearly independent eigenvector over $\mathbb{C}$.

$p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. eigenvalue 1,

$A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad m_{A_3}(x) = x^2+1 = (x-i)(x+i)$

$\det(x - A_3) = x^2+1 = (x-i)(x+i)$

2 linearly independent eigenvectors over $\mathbb{C}$.

$p = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\quad p_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

eigenvalue i $\quad$ eigenvalue $-i$

NO lin. indep. eigenvectors over $\mathbb{R}$.

$A_1 \oplus A_2 \oplus A_3 = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 1 & & \\ & & 0 & 1 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$

$m_A(x) = (x-1)^2(x-i)(x+i)$

$\det(x-A) = (x-1)^4(x-i)(x+i)$

lin. independent Eigenvectors

$p_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $p_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $p_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $p_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix}$ $p_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -i \end{pmatrix}$

Direct sums of subspaces

Let $\mathbb{F}$ be a field and V an $\mathbb{F}$-vector space.

$V = U \oplus W$ means

(a) U is a subspace of V

(b) W is a subspace of V.

(c) $V = U + W$

(d) $U \cap W = \{0\}$.

Here $U + W = \{u + w \mid u \in U \text{ and } w \in W\}$

$U \cap W = \{v \in V \mid v \in U \text{ and } v \in W\}$.

Proposition Let $\mathbb{F}$ be a field and V an $\mathbb{F}$-vector space. Assume

$$V = U \oplus W$$

Let B be a basis of U and

C a basis of V.

(a) Then $B \cup C$ is a basis of $U \oplus W$.

(b) Let $f_1 : U \to U$ be a linear transformation.

$f_2 : W \to W$ a linear transformation.

Let A_1 be the matrix of f_1 w.r.t. B

A_2 the matrix of f_2 w.r.t. C.

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Then $f : V \to V$

$u + w \mapsto f_1(u) + f_2(v)$

is (ba) a linear transformation
(bb) has matrix $A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right)$

w.r.t. the basis $\mathcal{B} \cup \mathcal{C}$.

(bc) $\ker f = \ker(f_1) \oplus \ker(f_2)$.

Prop