Jordan blocks

Let $d \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$.

The <u>Jordan block of size d and eigenvalue λ</u> is

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \quad \text{in } M_d(\mathbb{C})$$

<u>Theorem</u> (Jordan normal form). Let $\mathbb{C}$ be an algebraically closed field. Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$. Then there exists $P \in GL_n(\mathbb{C})$ such that

$P^{-1}AP$ is a direct sum of Jordan blocks.

The Jordan blocks for A are unique (up to reordering).

<u>Recall</u>: Direct sum of matrices.

$$A_1 \oplus A_2 = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$$

$$A_1 \oplus A_2 \oplus A_3 = \begin{pmatrix} \boxed{A_1} & & 0 \\ & \boxed{A_2} & \\ 0 & & \boxed{A_3} \end{pmatrix}$$

One Jordan blocks

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Example $A = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ is a Jordan block of size 2 with eigenvalue 5.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$Ae_1 = 5e_1$ and $Ae_2 = 5e_2 + e_1$.

$\det(x - A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}\right) = \det\begin{pmatrix} x-5 & -1 \\ 0 & x-5 \end{pmatrix}$

$= (x-5)^2$

$m_A(x) = (x-5)^2$, since $m_A(x)$ divides $\det(x-A)$

and $A - 5 = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so that $A - 5 \neq 0$.

Example Let $d \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$ and

$$A = J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}.$$

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $e_d = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Then

$Ae_1 = \lambda e_1$, $Ae_2 = \lambda e_2 + e_1$, $Ae_3 = \lambda e_3 + e_2, \ldots,$

$Ae_d = \lambda e_d + e_{d-1}$.

$\det(x - A) = (x - \lambda)^d$

$m_A(x) = (x - \lambda)^d$.

Direct sum of Jordan blocks with same eigenvalue.

Let $\lambda \in \mathbb{C}$ and $d_1, d_2, \dots, d_k \in \mathbb{C}$. Let

$$A = J_{d_1}(\lambda) \oplus \cdots \oplus J_{d_k}(\lambda) = \begin{pmatrix} \left.\boxed{\begin{matrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{matrix}}\right\} d_1 & & & 0 \\ & \left.\boxed{\begin{matrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{matrix}}\right\} d_2 & & \\ & & \ddots & \\ 0 & & & \left.\boxed{\begin{matrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{matrix}}\right\} d_k \end{pmatrix}.$$

Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith spot. Then

$Ae_1 = \lambda e_1$, $Ae_2 = \lambda e_2 + e_1$, $\dots$, $Ae_{d_1} = \lambda e_{d_1} - e_{d_1 - 1}$,

$Ae_{d_1+1} = \lambda e_{d_1+1}$, $Ae_{d_1+2} = \lambda e_{d_1+2} + e_{d_1+1}$, $\dots$, $Ae_{d_1+d_2} = \lambda e_{d_1+d_2} - e_{d_1+d_2-1}$, $\dots$

and
$$\det(x - A) = (x-\lambda)^{d_1 + \cdots + d_k}$$
$$m_A(x) = (x-\lambda)^{\max(d_1,\dots,d_k)}.$$

Recall: If $A_1 \in M_{n_1}(\mathbb{C})$ and $A_2 \in M_{n_2}(\mathbb{C})$ and

$$A = A_1 \oplus A_2 = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \text{ in } M_{n_1+n_2}(\mathbb{C})$$

then

$$\det(x-A) = \det(x-A_1)\det(x-A_2)$$
$$m_A(x) = \operatorname{lcm}(m_{A_1}(x), m_{A_2}(x)).$$

Note: Each Jordan block has a single eigenvector: these are

$e_1, e_{d_1+1}, e_{d_1+d_2+1}, \dots, e_{d_1+\cdots+d_{k-1}+1}$.

Jordan blocks with different eigenvalues

Let $\lambda \in \mathbb{C}$ and $d_1, \ldots, d_k \in \mathbb{Z}_{>0}$.

Let $\mu \in \mathbb{C}$ and $n_1, \ldots, n_s \in \mathbb{Z}_{>0}$. Let

$$A = J_{d_1}(\lambda) \oplus \cdots \oplus J_{d_k}(\lambda) \oplus J_{n_1}(\mu) \oplus \cdots \oplus J_{n_s}(\mu)$$

$$= \begin{pmatrix} \underbrace{\boxed{\begin{matrix} \lambda & 1 & 0 \\ 0 & \ddots & \lambda \end{matrix}}}_{d_1} & & & & 0 \\ & \ddots & & & \\ & & \underbrace{\boxed{\begin{matrix} \lambda & 1 & 0 \\ 0 & \ddots & \lambda \end{matrix}}}_{d_k} & & \\ & & & \underbrace{\boxed{\begin{matrix} \mu & 1 & 0 \\ 0 & \ddots & \mu \end{matrix}}}_{n_1} & \\ & & & & \ddots \\ 0 & & & & \underbrace{\boxed{\begin{matrix} \mu & 1 & 0 \\ 0 & \ddots & \mu \end{matrix}}}_{n_s} \end{pmatrix}$$

Then $\det(x - A) = (x-\lambda)^{d_1 + \cdots + d_k}(x-\mu)^{n_1 + \cdots + n_s}$

$m_A(x) = (x-\lambda)^{\max(d_1, \ldots, d_k)}(x-\mu)^{\max(n_1, \ldots, n_s)}$.

~~Prop~~ **Proposition** Let $A \in M_n(\mathbb{C})$. ~~Then there exist~~
A has Jordan form with blocks

$$J_{d_1}(\lambda), \ldots, J_{d_k}(\lambda), J_{n_1}(\mu), \ldots, J_{n_s}(\mu)$$

if and only if there is a basis

$$b_1^{\lambda,1}, \ldots, b_{d_1}^{\lambda,1}, b_1^{\lambda,2}, \ldots, b_{d_2}^{\lambda,2}, \ldots, b_1^{\lambda,k}, \ldots, b_{d_k}^{\lambda,k}$$

$$b_1^{\mu,1}, \ldots, b_{n_1}^{\mu,1}, b_1^{\mu,2}, \ldots, b_{n_2}^{\mu,2}, \ldots, b_1^{\mu,s}, \ldots, b_{n_s}^{\mu,s}$$

of $\mathbb{C}^n$ such that if $j \in \{1, \ldots, k\}$ then

$A b_1^{\lambda,j} = \lambda b_1^{\lambda,j}$ and $A b_i^{\lambda,j} = \lambda b_i^{\lambda,j} + b_{i-1}^{\lambda,j}$ for $i \in \{2, 3, \ldots, d_j\}$

and if $j \in \{1, \ldots, s\}$ then

$A b_1^{\mu,j} = \mu b_1^{\mu,j}$ and $A b_i^{\mu,j} = \mu b_i^{\mu,j} + b_{i-1}^{\mu,j}$ for $i \in \{2, 3, \ldots, n_j\}$.

If the vectors $b_1^{\lambda,1}, \dots, b_{n_s}^{\mu,s}$ for the columns of P

$$P = \begin{pmatrix} | & & | \\ b_1^{\lambda,1} & \cdots & b_{n_s}^{\mu,s} \\ | & & | \end{pmatrix}$$ then

$$P^{-1}AP = J_{d_1}(\lambda) \oplus \cdots \oplus J_{d_k}(\lambda) \oplus J_{n_1}(\mu) \oplus \cdots \oplus J_{n_s}(\mu)$$