27.08.2020
GTLA Lecture 1

<u>Jordan Normal Form</u> Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$. Then there exists $P \in GL_n(\mathbb{C})$ such that

$P^{-1}AP$ is a direct sum of Jordan blocks.

(Uniqueness statement: The sizes and the eigenvalues are the same no matter which P is used.)

<u>Definition</u> A is <u>semisimple</u>, or <u>diagonalisable</u> if there exists $P \in GL_n(\mathbb{C})$ such that

$P^{-1}AP$ is diagonal.

In other words, A is diagonalisable
if and only if
all Jordan blocks for A have size 1.

<u>Definition</u> A is <u>nilpotent</u> if there exists $k \in \mathbb{Z}_{>0}$ such that $A^k = 0$.

In other words, A is nilpotent
if and only if
all Jordan blocks for A have eigenvalue 0.

Proposition Let $A \in M_n(\mathbb{C})$. Let $P \in GL_n(\mathbb{C})$

(a) A is nilpotent if and only if
$P^{-1}AP$ is nilpotent.

(b) Let $d \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$. Let $J = J_d(\lambda)$.
J is nilpotent if and only if $\lambda = 0$.

Proof (a) $\Rightarrow$ Assume A is nilpotent.
To show: $P^{-1}AP$ is nilpotent.
To show: There exists $k \in \mathbb{Z}_{>0}$ such that
$(P^{-1}AP)^k = 0$.
Let $k \in \mathbb{Z}_{>0}$ be such that $A^k = 0$.
To show: $(P^{-1}AP)^k = 0$.

$$(P^{-1}AP)^k = \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{k\text{-times}}$$
$$= P^{-1}A^kP = P^{-1}0 \cdot P = 0.$$

(a) $\Leftarrow$ Assume $P^{-1}AP$ is nilpotent.
To show: A is nilpotent.
To show: There exists $k \in \mathbb{Z}_{>0}$ such that $A^k = 0$.
Let $k \in \mathbb{Z}_{>0}$ such that $(P^{-1}AP)^k = 0$.
To show: $A^k = 0$.

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$$A^k = (P(P^{-1}AP)P^{-1})^k$$

$$= P(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)P^{-1}$$

$$= P(P^{-1}AP)^k P^{-1} = P\cdot 0\cdot P^{-1} = 0.$$

So A is nilpotent.

(b) To show: (ba) If $\lambda = 0$ then $J_d(\lambda)$ is nilpotent.

(bb) If ~~λ~~ $\lambda \neq 0$ then $J_d(\lambda)$ is not nilpotent.

(ba) Assume $\lambda = 0$.

To show: $J_d(\lambda)$ is nilpotent.

To show: There exists $k \in \mathbb{Z}_{>0}$ such that $J_d(\lambda)^k = 0$.

Let $k = d$.

To show: $J_d(\lambda)^k = 0$.

$$J_d(\lambda)^k = J_d(0)^k = J_d(0)^d = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}^d$$

$$= \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}^{d-1} = \cdots = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}^2 = 0.$$

So $J_d(0)$ is nilpotent.

(bb) Assume $\lambda \neq 0$.

If $k \in \mathbb{Z}_{>0}$ ~~then~~, since $0 \neq 1 = (\lambda^{-1})^k \cdot \lambda^k$ then $\lambda^k \neq 0$.

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Since the $(1,1)$ entry of $J_d(\lambda)^k$ is λ^k

and $\lambda^k \neq 0$ then $J_d(\lambda)^k \neq 0$.

So $J_d(\lambda)$ is not nilpotent. //

<u>Theorem</u> (Jordan decomposition) Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$. Then there exist unique $S, N \in M_n(\mathbb{C})$ such that

(a) $A = S + N$,

(b) S is semisimple and N is nilpotent.

(c) $SN = NS$.

<u>Idea:</u> $P^{-1}AP$ is a direct sum of Jordan blocks. For each Jordan block in $P^{-1}AP$,

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$= S_\lambda + J_d(0).$$

Let C be the direct sum of the S_λ and X the direct sum of the $J_d(0)$.

Then $P^{-1}AP$ is the direct sum of $C + X$.

Let $S = PCP^{-1}$ and $N = PXP^{-1}$

Then $A = P(P^{-1}AP)P^{-1} = P(C+X)P^{-1} = PCP^{-1} + PXP^{-1} = S + N$.