A. Ram
01.09.2020 GTLA Lecture

<u>Theorem</u> (Block decomposition). Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Let

$$V = \mathbb{F}^n \quad \text{and} \quad A \in M_n(\mathbb{F}).$$

Let $m_A(x)$ be the minimal polynomial of A. Assume

$$m_A(x) = p(x)q(x) \quad \text{with} \quad \gcd(p(x), q(x)) = 1.$$

Let $r(x), s(x) \in \mathbb{F}[x]$ with

$$1 = p(x)r(x) + q(x)s(x)$$

Let $P_U = p(A)r(A)$ and $P_W = q(A)s(A)$,

$$U = P_U V \quad \text{and} \quad W = P_W V.$$

Then $P_U + P_W = 1$, $P_U^2 = P_U$, $P_W^2 = P_W$, $P_U P_W = 0$

and $\quad V = U \oplus W$

and U and W are both A-invariant.

<u>Theorem</u> (Jordan Normal Form). Let $A \in M_n(\mathbb{C})$. Then there exists $P \in GL_n(\mathbb{C})$ such that

$P^{-1}AP$ is a direct sum of Jordan Blocks.

A. Ram 01.09.2020

GTLA Lecture

Let $\mathbb{F}$ be a field, $n \in \mathbb{Z}_{>0}$.

A matrix $P \in M_n(\mathbb{C})$ is *invertible* if there exists $P^{-1} \in M_n(\mathbb{C})$ such that

$$PP^{-1} = 1 \quad \text{and} \quad P^{-1}P = 1.$$

The *general linear group* is

$$GL_n(\mathbb{F}) = \{P \in M_n(\mathbb{F}) \mid P \text{ is invertible}\}.$$

The *conjugation action* of $GL_n(\mathbb{F})$ on $M_n(\mathbb{F})$ is the function

$$\begin{aligned} GL_n(\mathbb{F}) \times M_n(\mathbb{F}) &\longrightarrow M_n(\mathbb{F}) \\ (P, A) &\longmapsto P^{-1}AP \end{aligned}$$

This is a motivation for:

Groups and Group actions

A *group* is a set G with a function

$$\begin{aligned} G \times G &\to G \\ (a,b) &\mapsto ab \end{aligned} \quad \text{such that}$$

(a) If $g_1, g_2, g_3 \in G$ then $(g_1 g_2) g_3 = g_1(g_2 g_3)$.

(b) There exist $1 \in G$ such that

if $g \in G$ then $1 \cdot g = g$ and $g \cdot 1 = g$

(c) If $g \in G$ then there exists $g^{-1} \in G$ such that

$gg^{-1} = 1$ and $g^{-1}g = 1$.

A. Ram 01.09.2020

Let G be a group and let S be a set.

An <u>action of G on S</u> is a function

$$\begin{array}{ccc} G\times S & \longrightarrow & S \\ (g,x) & \longmapsto & g\cdot x \end{array} \quad \text{such that}$$

(a) If $g_1, g_2 \in G$ and $x \in S$ then

$$g_1\cdot(g_2\cdot x) = (g_1 g_2)\cdot x.$$

(b) If $x \in S$ then $1\cdot x = x$.

A <u>subgroup</u> of G is a subset $H \subseteq G$ such that

(a) If $h_1, h_2 \in H$ then $h_1 h_2 \in H$,

(b) $1 \in H$,

(c) If $h \in H$ then $h^{-1} \in H$.

<u>Example</u> $SL_n(\mathbb{F}) = \{ P \in M_n(\mathbb{F}) \mid \det(P) = 1 \}$.

<u>Example</u> $S_n = \left\{ P \in M_n(\mathbb{F}) \;\middle|\; \begin{array}{l} P \text{ has exactly one } 1 \\ \text{in each row and each} \\ \text{column and all other} \\ \text{entries are } 0 \end{array} \right\}$

R. Ram 01.09.2020

GTLA Lecture (4)

Let G be a group, and let $g \in G$.

The <u>order</u> of G is the number of elements in G.

The <u>order</u> of g is the smallest $k \in \mathbb{Z}_{>0}$ such that $g^k = 1$.

<u>Examples</u>

$$S_3 = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} \right\}$$

$$S_2 = \left\{ \begin{pmatrix} 1&0\\0&1 \end{pmatrix}, \begin{pmatrix} 0&1\\1&0 \end{pmatrix} \right\}$$

$H \subseteq S_3$ given by $H = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} \right\}$

$K \subseteq S_3$ given by $K = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} \right\}$

<u>Multiplication tables</u>

$\cdot$	a	b
a	a	b
b	b	a

$H = \{a, b\}$.

$\cdot$	A	B
A	A	B
B	B	A

$K = \{A, B\}$

$+$	0	1
0	0	1
1	1	0

$\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$

$\cdot$	1	-1
1		
-1		

$\mu_2 = \{1, -1\}$