A. Ram 04.09.2020
GTLA Lecture (1)

The symmetric group S_6.

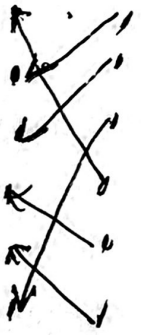
Compute $(12) \times (246) \times (123654)$.

$(12) =$  $=$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$(246) =$  $=$

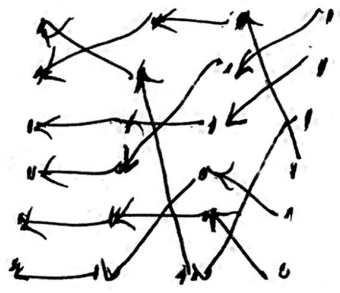
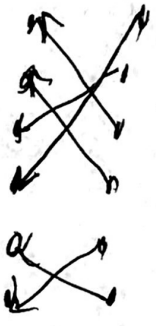
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$(123654) =$  $=$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Then

$(12) \times (246) \times (123654)$

$=$  $=$  $= (1423)(56)$

$$S_3 = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \right\}$$

$$= \{ \leftrightarrows, \leftrightarrows, \leftrightarrows, \leftrightarrows, \leftrightarrows, \leftrightarrows \}$$

$$= \{ 1, (12), (23), (13), (123), (132) \}$$

Let S be a subset of G.

The <u>subgroup of G generated by S</u> is the subgroup $H \subseteq G$ such that

(a) $H \supseteq S$,

(b) If K is a subgroup of G and $K \supseteq S$ then $K \supseteq H$.

In English: H is the smallest subgroups of G containing S.

Note: The form of this definition is similar to one of the characterisations of gcd and lcm. This is a "universal property".

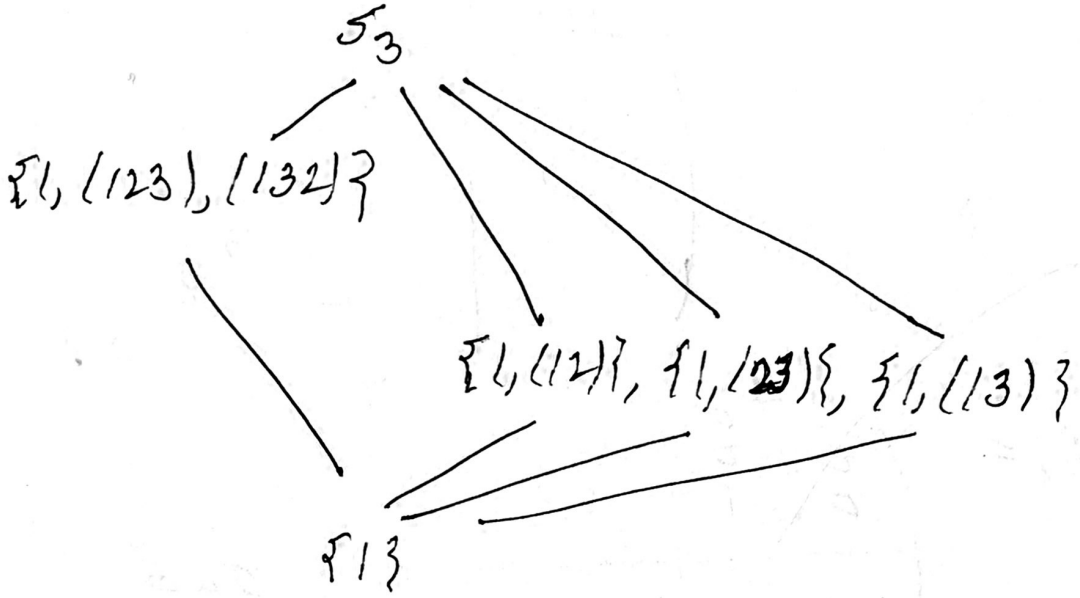
Let G be a group. Let $g \in G$.

The <u>order of g</u> is the smallest $k \in \mathbb{Z}_{>0}$ such that

$$\underbrace{g \heartsuit g \heartsuit g \heartsuit \cdots \heartsuit g}_{k\text{-times}} = \text{☺}$$

Subgroups of S_3

Cardinality



The group $\mathbb{Z}/10\mathbb{Z}$

$\mathbb{Z}/10\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$9+9+9+9+9+9+9+9+9+9 = 0$

$8+8+8+8+8 = 0$

$7+7+7+7+7+7+7+7+7+7 = 0$

$6+6+6+6+6 = 0$

$5+5 = 0$

$4+4+4+4+4 = 0$

$3+3+3+3+3+3+3+3+3+3 = 0$

$2+2+2+2+2 = 0$

$1+1+1+1+1+1+1+1+1+1 = 0$

$0 = 0$

order $|9| = 10$

order $|8| = 5$

order $|7| = 10$

order $|6| = 5$

order $|5| = 2$

order $|4| = 5$

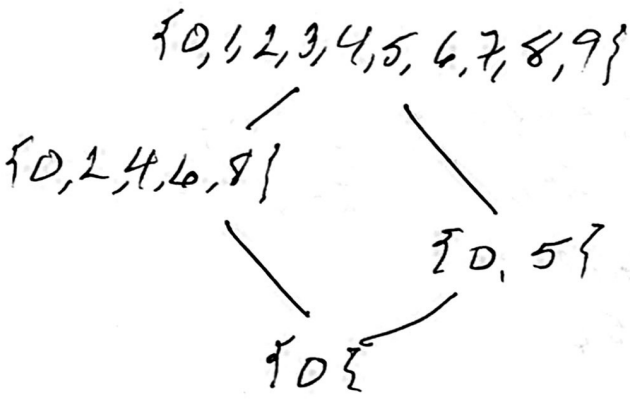
order $|3| = 10$

order $|2| = 5$

order $|1| = 10$

order $|0| = 1$.

Subgroups of $\mathbb{Z}/10\mathbb{Z}$



Cardinality
10
5
2
1

The subgroup generated by 3 is

$$\{0, 3, 3+3, 3+3+3, \ldots, 3+3+3+3+3+3+3+3+3+3\}$$

$$= \{0, 3, 6, 9, 2, 5, 8, 1, 4, 7\}$$

$$= \{0,1,2,3,4,5,6,7,8,9\} = \mathbb{Z}/10\mathbb{Z}$$

Proposition Let G be a group and let $g \in G$.

(a) Let $k \in \mathbb{Z}_{>0}$ and assume $\text{order}(g) = k$.
Then the subgroup generated by g is
$\{1, g, g^2, \ldots, g^{k-1}\}$ and
$\text{Card}\{1, g, g^2, \ldots, g^{k-1}\} = \text{order}(g)$

(b) Assume $\text{order}(g) = \infty$. Then the subgroup generated by g is

$$\{\ldots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \ldots\}$$

and
$\text{Card}\{\ldots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \ldots\} = \text{order}(g)$.

Let G be a group. A <u>cyclic subgroup of G</u> is a subgroup generated by one element.