08.09.2020
GTLA Lecture ①

Let G and H be groups.

The <u>direct product of G and H</u> is

$$G \times H = \{(g,h) \mid g \in G \text{ and } h \in H\}$$ with

$$(g_1,h_1)\cdot(g_2,h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$$

<u>Example</u> $\mu_2 \times \mu_2$ with $\mu_2 = \{1,-1\}$.

$\mu_2 \times \mu_2 = \{(1,1),(1,-1),(-1,1),(1,1)\}$ with multiplication table

$\cdot$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,1)$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,-1)$	$(1,-1)$	$(1,1)$	$(-1,-1)$	$(-1,1)$
$(-1,1)$	$(-1,1)$	$(-1,-1)$	$(1,1)$	$(1,-1)$
$(-1,-1)$	$(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$

order$(1,1) = 1$ order$(-1,1) = 2$
order$(1,-1) = 2$ order$(-1,-1) = 2$

<u>Example</u> $\mu_4 = \{1, i, -1, -i\} = \{1, i, i^2, i^3\}$, $\mu_4 \cong \mathbb{Z}/4\mathbb{Z}$

order$(1) = 1$ order$(-1) = 2$
order$(i) = 4$ order$(-i) = 4$

So $\mu_2 \times \mu_2$ is <u>not</u> isomorphic to $\mu_4 \cong \mathbb{Z}/4\mathbb{Z}$.

Let G, H be groups.

A <u>homomorphism from G to H</u> is a function $f: G \to H$ such that

(a) if $g_1, g_2 \in G$ then $f(g_1 g_2) = f(g_1) \heartsuit f(g_2)$

(b) $f(1) = ☺$

(c) If $g \in G$ then $f(g^{-1}) = (g)f$

Let $f: G \to H$ be a homomorphism.

$\ker f = \{ g \in G \mid f(g) = ☺ \}$

$\operatorname{im} f = \{ f(g) \mid g \in G \}$.

A <u>normal subgroup of G</u> is a subgroup K such that

if $n \in K$ and $g \in G$ then $g^{-1} n g \in K$.

<u>Proposition</u> Let $f: G \to H$ be a homomorphism.

(a) $\ker f$ is a normal subgroup of G

(b) $\operatorname{im} f$ is ~~a~~ a subgroup of H

<u>Proof</u> (a)

To show: (aa) If $n_1, n_2 \in \ker f$ then $n_1 n_2 \in \ker f$.

(ab) $1 \in \ker f$

(ac) If $n \in \ker f$ then $n^{-1} \in \ker f$

(ad) If $n \in \ker f$ and $g \in G$ then $g^{-1} n g \in \ker f$

(aa) Assume $n_1, n_2 \in \ker f$

To show: $n_1 n_2 \in \ker f$

To show: $f(n_1 n_2) = 1$.

Since $n_1, n_2 \in \ker f$ then

$$f(n_1 n_2) = f(n_1) f(n_2) = 1 \cdot 1 = 1$$

So $n_1 n_2 \in \ker f$.

(ab) To show: $1 \in \ker f$

To show: $f(1) = 1$.

By property (b) in the definition of homomorphism, $f(1) = 1$.

So $1 \in \ker f$

(ac) To show: If $n \in \ker f$ then $n^{-1} \in \ker f$.

Assume $n \in \ker f$

To show: $n^{-1} \in \ker f$

To show: $f(n^{-1}) = 1$.

Since $n \in \ker f$ then $f(n) = 1$ and so

$$f(n^{-1}) = f(n)^{-1} = 1^{-1} = 1,$$

where the first equality is by property (c) in the definition of homomorphism.

(ad) To show: If $n \in \ker f$ and $g \in G$ then $g^{-1} n g \in \ker f$.

Assume $n \in \ker f$ and $g \in G$.

To show: $g^{-1} n g \in \ker f$

To show: $f(g^{-1}ng) = 1$.

Since $n \in \ker f$ then $f(n) = 1$ and so

$$f(g^{-1}ng) = f(g^{-1}) f(n) f(g)$$
$$= f(g)^{-1} f(n) f(g) = f(g)^{-1} 1 \cdot f(g)$$
$$= f(g)^{-1} f(g) = 1.$$

So $\ker f$ is a normal subgroup of G.

(b) To show: $\mathrm{im} f$ is a subgroup of H.

To show: (ba) If $h_1, h_2 \in \mathrm{im}(f)$ then $h_1 h_2 \in \mathrm{im}(f)$.

(bb) $1 \in \mathrm{im}(f)$

(bc) If $h \in \mathrm{im}(f)$ then $h^{-1} \in \mathrm{im}(f)$.

(ba) Assume $h_1, h_2 \in \mathrm{im}(f)$

Then there exist $g_1, g_2 \in G$ such that

$f(g_1) = h_1$ and $f(g_2) = h_2$.

To show: $h_1 h_2 \in \mathrm{im}(f)$.

To show: There exists $g \in G$ such that $f(g) = h_1 h_2$

Let $g = g_1 g_2$

To show: $f(g) = h_1 h_2$.

$$f(g) = f(g_1 g_2) = f(g_1) f(g_2) = h_1 h_2.$$

So $h_1 h_2 \in \mathrm{im}(f)$.

(6b) To show: $1 \in \mathrm{im}(f)$.

Since $f(1) = 1$ then $1 \in \mathrm{im}(f)$.

(6c) ~~To~~ show: If $h \in \mathrm{im} f$ then $h^{-1} \in \mathrm{im} f$.

Assume $h \in \mathrm{im} f$.

Then ~~To show:~~ there exists $g \in G$ such that $f(g) = h$.

To show: $h^{-1} \in \mathrm{im} f$.

Since $f(g^{-1}) = f(g)^{-1} = h^{-1}$ then $h^{-1} \in \mathrm{im} f$.

So $\mathrm{im} f$ is a subgroup of H. //.