Cosets

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GTLA Lecture

Let G be a group. Let H be a subgroup.

The <u>set of cosets of H in G</u> is

$$G/H = \{gH \mid g \in G\}$$

where a <u>coset of H in G</u> is

$$gH = \{gh \mid h \in H\}.$$

<u>Example</u> $S_3 = G = \{1, r, r^2, s, sr, sr^2\}$ with $r^3 = 1$, $s^2 = 1$, $rs = sr^{-1}$ and $H = \{1, r, r^2\}$.

Then

$$1 \cdot H = \{1, r, r^2\} \qquad sH = \{s, sr, sr^2\}$$
$$r \cdot H = \{r, r^2, 1\} \qquad srH = \{sr, sr^2, s\}$$
$$r^2 \cdot H = \{r^2, r, 1\} \qquad sr^2H = \{sr^2, s, sr\}$$

and

$$G/H = \{H, sH\} \quad \text{and} \quad \mathrm{Card}(G/H) = 2.$$

<u>Theorem</u> Let G be a group and H a subgroup.

(a) The cosets partition the group G.

(b) If $g \in G$ then $\mathrm{Card}(gH) = \mathrm{Card}(H)$.

(c) $\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H)$.

Proof of (b)

To show: If $g \in G$ then $\mathrm{Card}(gH) = \mathrm{Card}(H)$.

Assume $g \in G$.

To show: There exists a bijection $\varphi : H \to gH$.

Let

$$\begin{matrix} \varphi : H \to gH \\ h \mapsto gh \end{matrix} \quad \text{and} \quad \begin{matrix} \psi : gH \to H \\ x \mapsto g^{-1}x \end{matrix}$$

To show: φ is a bijection.

To show: ψ is an inverse function to φ.

Since

$$(\psi \circ \varphi)(h) = \psi(\varphi(h)) = \psi(gh) = g^{-1}gh = h$$

and

$$(\varphi \circ \psi)(x) = \varphi(\psi(x)) = \varphi(g^{-1}x) = gg^{-1}x = x$$

then φ and ψ are inverse functions.

So φ is a bijection.

So $\mathrm{Card}(H) = \mathrm{Card}(gH)$.

Proof of (a)

To show: (aa) $\displaystyle\bigcup_{g \in G} gH = G$

(ab) If $g_1, g_2 \in G$ and $g_1H \cap g_2H \neq \emptyset$ then $g_1H = g_2H$.

(aa) To show: (aaa) $\bigcup_{g\in G} gH \subseteq G$

(aab) $G \subseteq \bigcup_{g\in G} gH$.

(aaa) Since $gH = \{gh \mid h\in H\} \subseteq G$ then

$\bigcup_{g\in G} gH \subseteq G$.

(aab) To show: If $k\in G$ then there exists $g\in G$ such that $k\in gH$.

Assume $k\in G$.

Let $g=k$. Then

$k = g = g\cdot 1 \in gH$.

So $G \subseteq \bigcup_{g\in G} gH$ and $G = \bigcup_{g\in G} gH$

(ab) To show: If $g_1, g_2 \in G$ and $g_1H \cap g_2H \neq \emptyset$ then $g_1H = g_2H$.

Assume $g_1, g_2\in G$ and $g_1H\cap g_2H \neq \emptyset$.

Let $z \in g_1H \cap g_2H$.

Then there exist $h_1, h_2 \in H$ such that

$z = g_1h_1$ and $z = g_2h_2$

So $g_1 = zh_1^{-1} = g_2h_2h_1^{-1}$ and $g_2 = zh_2^{-1} = g_1h_1h_2^{-1}$

To show: $g_1H = g_2H$.

To show: (aba) $g_1H \subseteq g_2H$

(abb) $g_2H \subseteq g_1H$

(aba) To show: If $a \in g_1H$ then $a \in g_2H$.

Assume $a \in g_1H$.

Then there exists $h \in H$ such that $a = g_1h$.

So $a = g_1h = g_2h_2h_1^{-1}h = g_2(h_2h_1^{-1}h) \in g_2H$.

So $g_1H \subseteq g_2H$

(abb) To show: If $b \in g_2H$ then $b \in g_1H$.

Assume $b \in g_2H$

Then there exists $h' \in H$ such that $b = g_2h'$.

So $b = g_2h' = g_1h_1h_2^{-1}h' = g_1(h_1h_2^{-1}h') \in g_1H$.

So $g_2H \subseteq g_1H$ and $g_1H = g_2H$.

(c) To show: $\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H)$

Since $G = \bigcup_{gH \in G/H} gH$ and $\mathrm{Card}(gH) = \mathrm{Card}(H)$

then

$$\mathrm{Card}(G) = \sum_{gH \in G/H} \mathrm{Card}(gH) = \sum_{gH \in G/H} \mathrm{Card}(H)$$

$$= \mathrm{Card}(H)\Big(\sum_{gH \in G/H} 1\Big) = \mathrm{Card}(H) \cdot \mathrm{Card}(G/H).$$