3. <u>First Sylow Theorem</u> Let G be a finite group. Let $p \in \mathbb{Z}_{>0}$ be a prime. Write

$$\mathrm{Card}(G) = p^a b,$$

where b is not divisible by p. A <u>p-Sylow subgroup</u> of G is a subgroup of G of cardinality p^a. Show that G has a p-Sylow subgroup.

3a) Let $\Lambda^{p^a}(G)$ be the set of subsets of G of cardinality p^a. Show that if $j \in \{1, \dots, p^a\}$ and p^i divides $p^a b - j$ then p^i divides $p^a - j$.

Conclude that

$$\operatorname{Card}(\Lambda^{p^a}(G)) = \binom{p^a b}{p^a}$$ is not divisible by p.

3b) Consider the action of G on $\Lambda^{p^a}(G)$ by left multiplication and use

$$\mathrm{Card}(\Lambda^{p^a}(G)) = \sum_{\substack{\text{distinct} \\ \text{orbits}}} \mathrm{Card}(GS)$$

to conclude that there exists $S \in \Lambda^{p^a}(G)$ such that the cardinality of the orbit of S is not divisible by p.

$\underline{3c)}$ Let $P = Stab_G(S)$ and show that $Card(P) = p^a$.

4. <u>Second Sylow Theorem</u> Let G be a finite group. Let $p \in \mathbb{Z}_{\geq 0}$ be a prime. Write

$$\mathrm{Card}(G) = p^a b,$$

where b is not divisible by p. A <u>p-Sylow subgroup of G</u> is a subgroup of G of cardinality p^a.

Show that all p-Sylow subgroups of G are conjugate.

4a) Let P and H be p-Sylow subgroups of G.
Let H act on G/P by left multiplication.
Use

$$\mathrm{Card}(G/P) = \sum_{\substack{\text{distinct} \\ \text{orbits}}}' \mathrm{Card}(HgP).$$

to show that there is an orbit HgP with $\mathrm{Card}(HgP) = 1$.

4b) Show that $H \subseteq gPg^{-1}$ and conclude that $H = gPg^{-1}$!

5) <u>Third Sylow Theorem</u> Let G be a finite group. Let $p \in \mathbb{Z}_{>0}$ be prime. Write

$$\mathrm{Card}(G) = p^a b,$$

where b is not divisible by p. A <u>p-Sylow subgroup</u> of G is a subgroup of G of cardinality p^a. Show that the number of p-Sylow subgroups of G is $1 \bmod p$.

5a) Let P be a p-Sylow subgroup of G.
Let P act on the set S of p-Sylow subgroups of G by conjugation. Show that if
$P*Q$ is an orbit under this action
then $\mathrm{Card}(P*Q)=1$ or p divides $\mathrm{Card}(P*Q)$.

5b) Assume Card$(P \cap Q) = 1$ and let $N(Q)$ be the normalizer of Q.
Show that P and Q are both p-Sylow subgroups of $N(Q)$.

5c) Assume Card $(P \cap Q) = 1$. Use the second Sylow theorem and part (b) to show that $P = Q$.

5d) Use part (a) and (c) and

$$\mathrm{Card}(S) = \sum_{\substack{\text{distinct} \\ \text{orbits}}} \mathrm{Card}(P * Q).$$

to conclude that $\mathrm{Card}(S) \equiv 1 \mod p$.

6) <u>Fourth Sylow theorem</u> Let G be a finite group. Let $p \in \mathbb{Z}_{>0}$ be a prime. Write

$$\mathrm{Card}(G) = p^a b,$$ where b is not divisible by p. A <u>p-Sylow subgroup</u> of G is a subgroup of G of cardinality p^a. Show the number of p-Sylow subgroups divides $\mathrm{Card}(G)$.

6a) Let G act on the set $\mathcal{P}$ of p-Sylow subgroups of G by conjugation. Use the second Sylow theorem to conclude that there is only one orbit under this action.

6b) Conclude from (a) that the number of p-Sylow subgroups of G divides Card(G).