17.09.2020
GTLA Lecture 1

Let $f: G \to H$ be a group homomorphism.
Let
$$K = \ker f \text{ and } I = \operatorname{im} f.$$
We have proved

(a) I is a subgroup of H

(b) K is a normal subgroup of G.

(c) G/K with $\begin{array}{ccc} G/K \times G/K & \to & G/K \\ (aK, bK) & \mapsto & abK \end{array}$ is a group.

<u>Theorem</u> $G/_{\ker f} \cong \operatorname{im} f.$

<u>Proof</u> To show: there exists an isomorphism
$$\varphi: G/K \to I.$$
Let
$$\begin{array}{rccc} \varphi: & G/K & \to & I. \\ & gK & \mapsto & f(g). \end{array}$$
To show: (a) φ is a function.

(b) φ is a homomorphism.

(c) φ is bijective.

(a) To show: If $g_1, g_2 \in G$ and $g_1K = g_2K$ then
$$f(g_1) = f(g_2).$$

Assume $g_1, g_2 \in G$ and $g_1K = g_2K$.
Since $g_1 \in g_2K$ then there exists $k \in K$ such that $g_1 = g_2k$.

So
$$f(g_1) = f(g_2k) = f(g_2)f(k)$$
$$= f(g_2)\cdot 1 \quad (\text{since } k \in \ker f)$$
$$= f(g_2).$$
So φ is a function.

(b) To show: φ is a homomorphism.
To show: If $g_1, g_2 \in G$ then $\varphi(g_1K \cdot g_2K) = \varphi(g_1K)\varphi(g_2K)$.

Assume $g_1, g_2 \in G$. Then
$$\varphi(g_1K \cdot g_2K) = \varphi(g_1g_2K) = f(g_1g_2) \text{ and}$$
$$\varphi(g_1K)\varphi(g_2K) = f(g_1)f(g_2) \text{ and}$$
$f(g_1g_2) = f(g_1)f(g_2)$ since f is a homomorphism.
So φ is a homomorphism.

(c) To show: (ca) φ is injective
(cb) φ is surjective.

(ca) To show: If $g_1, g_2 \in G$ and $\varphi(g_1K) = \varphi(g_2K)$
then $g_1K = g_2K$.

Assume $g_1, g_2 \in G$ and $\varphi(g_1K) = \varphi(g_2K)$.

Then $f(g_1) = f(g_2)$

So $1 = f(g_1)^{-1} f(g_2) = f(g_1^{-1}) f(g_2) = f(g_1^{-1} g_2)$.

So $g_1^{-1} g_2 \in \ker f = K$.

So $g_2 \in g_1 K$.

So $g_2 K \cap g_1 K \neq \emptyset$

So $g_1 K = g_2 K$. (since the cosets of K in G partition G.)

So φ is injective.

(cb) To show: If $l \in I$ then there exists $g \in G$ such that $\varphi(gK) = l$.

Assume $l \in I = \mathrm{im} f$.

Then there exists $g \in G$ such that $f(g) = l$.

Then $\varphi(gK) = f(g) = l$.

So φ is surjective.

So $\varphi: G/K \to I$ is a bijective homomorphism.

So $G/K \cong I$. So $G/\ker f \cong \mathrm{im} f$.

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Centralizers and conjugacy classes

Let G be a group and let $x \in G$.

The centralizer of x is

$$Z_G(x) = \{ g \in G \mid gxg^{-1} = x \}$$

The conjugacy class of x is

$$\mathcal{C}_x = \{ gxg^{-1} \mid g \in G \}$$

Let G be a group and let S be a set.

An action of G on S is a function

$$\begin{aligned} G \times S &\to S \\ (g, x) &\mapsto g \cdot x \end{aligned}$$ such that

(a) If $g_1, g_2 \in G$ and $x \in S$ then

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x.$$

(b) If $x \in S$ then $1 \cdot x = x$.

Let $x \in S$. The stabilizer of x is

$$\mathrm{Stab}_G(x) = \{ g \in G \mid g \cdot x = x \}$$

The orbit of x is

$$G \cdot x = \{ g \cdot x \mid g \in G \}.$$

Let G be a group.

The <u>conjugation action of G on G</u> is

$$G \times G \longrightarrow G$$

$$(g, x) \mapsto g \diamond x \quad \text{where} \quad g \diamond x = g x g^{-1}.$$

This is an action since

$$g_1 \diamond (g_2 \diamond x) = g_1 \diamond (g_2 x g_2^{-1}) = g_1 (g_2 x g_2^{-1}) g_1^{-1}$$

$$= (g_1 g_2) x (g_2^{-1} g_1^{-1}) = (g_1 g_2) x (g_1 g_2)^{-1}$$

$$= (g_1 g_2) \diamond x, \quad \text{and}$$

$$1 \diamond x = 1 x 1^{-1} = 1 \cdot x \cdot 1 = x.$$

For the conjugation action of G on G

$\mathrm{Stab}_G(x) = Z_G(x)$, the centralizer of x, and

$G \diamond x = \mathcal{C}_x$, the conjugacy class of x.