Let G be a group.

A <u>Gset</u> is a set S with an action of G on S.

An <u>action of G on S</u> is a function

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, x) &\longmapsto g \cdot x \end{aligned} \quad \text{such that}$$

(a) If $g_1, g_2 \in G$ and $x \in S$ then $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

(b) If $x \in S$ then $1 \cdot x = x$

Let S be a G-set. Let $x \in S$.

$$\mathrm{Stab}_G(x) = \{ g \in G \mid g \cdot x = x \}.$$

$$G \cdot x = \{ g \cdot x \mid g \in G \}.$$

<u>Theorem</u> Let S be a G-set.

(a) The orbits partition S

(b) Let $x \in S$ and let $H = \mathrm{Stab}_G(x)$. Then

$$\mathrm{Card}(G/H) = \mathrm{Card}(G \cdot x)$$

(c) Let $x \in S$. Then

$$\mathrm{Card}(G) = \mathrm{Card}(G \cdot x)\, \mathrm{Card}(\mathrm{Stab}_G(x)).$$

(d) Let $x \in S$ and let $g \in G$. Then

$$\mathrm{Stab}_G(g \cdot x) = g\, \mathrm{Stab}_G(x)\, g^{-1}.$$

Proof of (b) To show: There exists a bijection

$$\varphi : G/H \to G\cdot x.$$

Let $\varphi : G/H \to G\cdot x$
$\quad gH \mapsto gx.$

To show: (ba) φ is a function
(bb) φ is injective
(bc) φ is surjective.

(ba) To show: If $g_1, g_2 \in G$ and $g_1 H = g_2 H$
then $\varphi(g_1 H) = \varphi(g_2 H)$

Assume $g_1, g_2 \in G$ and $g_1 H = g_2 H$.

Since $g_1 = g_1 \cdot 1 \in g_1 H = g_2 H$ then there exists
$h \in H$ such that $g_1 = g_2 h$.

To show: $\varphi(g_1 H) = \varphi(g_2 H)$.

$$\varphi(g_1 H) = g_1 x = g_2 h x = g_2 x = \varphi(g_2 H)$$

since $h \in H = \mathrm{Stab}_G(x)$.

So φ is a function.

(bb) To show: φ is injective.

To show: If $g_1, g_2 \in G$ and $\varphi(g_1H) = \varphi(g_2H)$
then $g_1H = g_2H$.

Assume $g_1, g_2 \in G$ and $\varphi(g_1H) = \varphi(g_2H)$.

Then $g_1x = g_2x$.

So $x = g_1^{-1}g_2x$. and $g_1^{-1}g_2 \in H$.

So
$$g_2 = g_1g_1^{-1}g_2 = g_1(g_1^{-1}g_2) \in g_1H.$$

So $g_2H \cap g_1H \neq \emptyset$.

So $g_1H = g_2H$.

(bc) To show: φ is surjective.

To show: If $y \in G \cdot x$ then there exists $g \in G$ such that $\varphi(gH) = y$.

Assume $y \in G \cdot x$.

Then there exists $g \in G$ such that $y = g \cdot x$

Then $\varphi(gH) = g \cdot x = y$.

So φ is surjective

So φ is bijective.

So $\mathrm{Card}(G/H) = \mathrm{Card}(G \cdot x)$.

(c) To show: $\mathrm{Card}(G) = \mathrm{Card}(G \cdot x)\,\mathrm{Card}(\mathrm{Stab}_G(x))$.

Let $H = \mathrm{Stab}_G(x)$. Then

$$\begin{aligned}\mathrm{Card}(G) &= \mathrm{Card}(G/H)\,\mathrm{Card}(H)\\ &= \mathrm{Card}(G \cdot x)\,\mathrm{Card}(H)\\ &= \mathrm{Card}(G \cdot x)\,\mathrm{Card}(\mathrm{Stab}_G(x)).\end{aligned}$$

(d) To show: If $g \in G$ and $x \in S$ then

$$\mathrm{Stab}_G(g \cdot x) = g\,\mathrm{Stab}_G(x)\,g^{-1}.$$

Assume $g \in G$ and $x \in S$.

To show: (da) $\mathrm{Stab}_G(gx) \subseteq g\,\mathrm{Stab}_G(x)\,g^{-1}$

(db) $g\,\mathrm{Stab}_G(x)\,g^{-1} \subseteq \mathrm{Stab}_G(gx)$.

(da) Let $y \in \mathrm{Stab}_G(gx)$

To show: There exists $h \in \mathrm{Stab}_G(x)$ such that $y = ghg^{-1}$.

Let $h = g^{-1}yg$.

Then $h \cdot x = g^{-1}ygx = g^{-1}gx = 1 \cdot x = x$.

So $h \in \mathrm{Stab}_G(x)$ and $ghg^{-1} = gg^{-1}ygg^{-1} = y$.

(db) To show: $g\,\mathrm{Stab}_G(x)\,g^{-1} \subseteq \mathrm{Stab}_G(gx)$.

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Let $y \in g\,\mathrm{Stab}_G(x)g^{-1}$.

Then there exists $h \in \mathrm{Stab}_G(x)$ such that

$$y = ghg^{-1}.$$

To show: $y \in \mathrm{Stab}_G(g \cdot x)$.

Since

$$y \cdot \cancel{gx} = ghg^{-1}gx = ghx = gx$$

then $y \in \mathrm{Stab}_G(g \cdot x)$.

So $\mathrm{Stab}_G(x) = \mathrm{Stab}_G(g \cdot x)$.