22.09.2020
GTLA Lecture 4

The conjugation action of G on G

Let G be a group.
The <u>conjugation action of G on G</u> is

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto g \diamond x \quad \text{where } g \diamond x = g x g^{-1}.$$

Let $x \in G$. The <u>centralizer of x</u> is

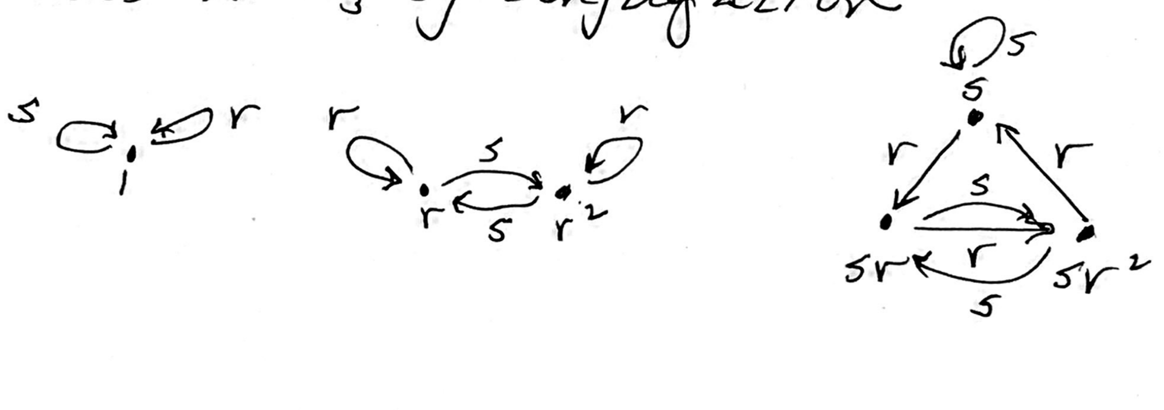
$$Z_G(x) = \{ g \in G \mid g x g^{-1} = x \}.$$

This is the same as the stabilizer of x under the conjugation action of G on G.
The <u>conjugacy class of x</u> is

$$\mathcal{C}_x = \{ g x g^{-1} \mid g \in G \}.$$

This is the same as the <u>orbit</u> of x under the conjugation action of G on G.

<u>Example</u> $S_3 = \{1, r, r^2, s, sr, sr^2\}$ with
$r^3 = 1$, $s^2 = 1$ and $rs = sr^{-1}$.
S_3 acts on S_3 by conjugation



22.09.2020

GTA lecture (2)

since

$r1r^{-1}=1,$ $\quad$ $s1s^{-1}=1,$

$rrr^{-1}=r,$ $\quad$ $srs^{-1}=srs=s^2r^2=r^2,$

$rr^2r^{-1}=r^2,$ $\quad$ $sr^2s^{-1}=sr^2s=srsr^2$

$$=s^2r^2r^2=r^4=r,$$

and $rsr^{-1}=rsr^2=sr^2r^2=sr^4=sr,$

$sss^{-1}=s,$

$r(sr)r^{-1}=rs=sr^2,$

$s(sr)s^{-1}=s^2rs=s^2sr^2=sr^2,$

$r(sr^2)r^{-1}=rsr^2r^2=rsr=sr^2r=sr^3=s,$

$s(sr^2)s^{-1}=s^2r^2s=r^2s=rsr^2=sr^2r^2=sr^4=sr.$

So $C_1=\{1\},$ $\quad C_r=\{r,r^2\},$ $\quad C_s=\{s,sr,sr^2\}$

$Z_G(1)=\{1,r,r^2,s,sr,sr^2\}$

$Z_G(r)=\{1,r,r^2\},$ $\quad Z_G(r^2)=\{1,r,r^2\}$

$Z_G(s)=\{1,s\},$ $\quad Z_G(sr)=\{sr,1\},$ $\quad Z_G(sr^2)=\{1,sr^2\}.$

The <u>center of G</u> is

$$Z(G) = \{ z \in G \mid \text{if } g \in G \text{ then } zg = gz \}.$$

Then

$$Z(G) = \{ z \in G \mid \text{if } g \in G \text{ then } gzg^{-1} = z \}$$

$$= \{ z \in G \mid \mathrm{Stab}_G(z) = G \}$$

$$= \{ z \in G \mid G \diamond z = \{z\} \}$$

So $Z(G)$ is the union of the conjugacy classes of size 1.

<u>Note:</u> If $z \in Z(G)$ and $g \in G$ then

$$gzg^{-1} = z \in G.$$

So $Z(G)$ is a normal subgroup of G

(If $z_1, z_2 \in G$ and $g \in G$ then $g z_1 z_2 g^{-1}$
$= g z_1 g^{-1} g z_2 g^{-1} = z_1 z_2$ and $z_1 z_2 \in Z(G)$).

If S is a G-set then the orbits of partition S. So

$$\mathrm{Card}(S) = \sum_{\substack{\text{distinct} \\ \text{orbits } G \cdot x_i}} \mathrm{Card}(G \cdot x_i).$$

Apply this to the action of G on G by conjugation.

$$\mathrm{Card}(G) = \sum_{\substack{\text{distinct}\\ \text{conjugacy}\\ \text{classes } \mathcal{C}_{x_i}}} \mathrm{Card}(\mathcal{C}_{x_i}).$$

Since

$$Z(G) = \bigcup_{\substack{\text{conjugacy}\\ \text{classes of}\\ \text{size } 1}} \mathcal{C}_z \quad \text{then}$$

$$\mathrm{Card}(G) = \mathrm{Card}(Z(G)) + \sum_{\substack{\text{conj. classes}\\ \text{with}\\ \mathrm{Card}(\mathcal{C}_{x_i}) > 1}} \mathrm{Card}(\mathcal{C}_{x_i}).$$

This is the <u>class</u> <u>equation</u>.

For our example:

$G = S_3 = \{1, r, r^2, s, sr, sr^2\}$ then

$Z(G) = \mathcal{C}_1 = \{1\}$

And the class equation is

$$\mathrm{Card}(G) = \mathrm{Card}(Z(G)) + \sum_{\substack{\text{conj. classes}\\ \text{with}\\ \mathrm{Card}(\mathcal{C}_{x_i}) > 1}} \mathrm{Card}(\mathcal{C}_{x_i})$$

So

$$6 = 1 + (2 + 3).$$

The <u>standard inner product on $\mathbb{R}^n$</u> is

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(\vec{u}, \vec{v}) \longmapsto \vec{u} \cdot \vec{v}$$ given by

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

The <u>standard inner product on $\mathbb{C}^n$</u> is

$$\mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$
$$(\vec{u}, \vec{v}) \longmapsto \vec{u} \cdot \vec{v}$$ given by

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 \overline{v_1} + u_2 \overline{v_2} + \cdots + u_n \overline{v_n}$$

where $\overline{x+iy} = x - iy$ is the conjugate of $x+iy \in \mathbb{C}$ (where $x, y \in \mathbb{R}$ and $i^2 = -1$).