Let $\mathbb{F}$ be a field and let $^{-}:\mathbb{F}\to\mathbb{F}$ be a function that satisfies

if $c_1, c_2\in\mathbb{F}$ then $\bar{1}=1$, $\overline{c_1+c_2}=\bar{c}_1+\bar{c}_2$ and

$$\overline{c_1c_2}=\bar{c}_2\cdot\bar{c}_1.$$

The favorite examples are

(1) $\mathbb{F}=\mathbb{C}$ and $^{-}:\mathbb{C}\to\mathbb{C}$ is complex conj.

(2) $\mathbb{F}$ is any field $^{-}:\mathbb{C}\to\mathbb{C}$ is the identity

Let V be an $\mathbb{F}$-vector space.

A <u>sesquilinear form on V</u> is a function

$$\begin{array}{rcl} V\times V & \to & \mathbb{F} \\ (x,y) & \mapsto & \langle x,y\rangle \end{array} \quad \text{such that}$$

(a) If $x_1,x_2,y\in V$ then $\langle x_1+x_2,y\rangle=\langle x_1,y\rangle+\langle x_2,y\rangle$.

(b) If $x,y_1,y_2\in V$ then $\langle x,y_1+y_2\rangle=\langle x,y_1\rangle+\langle x,y_2\rangle$.

(c) If $c\in\mathbb{F}$ and $x,y\in V$ then $\langle cx,y\rangle=c\langle x,y\rangle$.

(d) If $c\in\mathbb{F}$ and $x,y\in V$ then $\langle x,cy\rangle=\bar{c}\langle x,y\rangle$.

A sesquilinear form is <u>Hermitian</u> if it satisfies:

If $x,y\in V$ then $\langle y,x\rangle=\overline{\langle x,y\rangle}$.

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis of V.

The Gram matrix of $\langle , \rangle$ with respect to $\mathcal{B}$ is the matrix $G_{\mathcal{B}} \in M_n(\mathbb{F})$ given by

$$G_{\mathcal{B}}(i,j) = \langle b_i, b_j \rangle.$$

Let $W \subseteq V$ be a subspace of V.

The orthogonal to W is

$$W^{\perp} = \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w \rangle = 0\}.$$

The subspace W is nonisotropic if $W \cap W^{\perp} = 0$.

Proposition Let $\langle\, \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form. Then $\langle , \rangle$ satisfies

if $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$

if and only if $\langle , \rangle$ satisfies

if $W \subseteq V$ is a subspace then $W \cap W^{\perp} = 0$.

Proof $\Rightarrow$ Assume $W \subseteq V$ is a subspace.

To show: If $w \in W \cap W^{\perp}$ then $w = 0$.

Assume $w \in W \cap W^{\perp}$.

Then $\langle w, w \rangle = 0$. So $w = 0$.

$\Leftarrow$ Assume $v \in V$ and $\langle v, v \rangle = 0$.

To show: $v = 0$.

Let $W = \mathrm{span}\{v\}$.

Since $\langle v, v \rangle = 0$ then $v \in W \cap W^\perp$.

Since $W \cap W^\perp = 0$ then $v = 0$. //.

Let $W \subseteq V$ be a subspace.

Assume $k \in \mathbb{Z}_{>0}$ and $\{w_1, \ldots, w_k\}$ is a basis of W.

A <u>dual</u> <u>basis</u> <u>to</u> $\{w_1, \ldots, w_k\}$ is a basis $\{w^1, \ldots, w^k\}$ of W such that

$$\langle w^i, w_j \rangle = \delta_{ij}.$$

Let $C = \{w_1, \ldots, w_k\}$.

<u>Proposition</u> The following are equivalent:

(a) $\{w^1, \ldots, w^k\}$ exists

(b) G_C is invertible

(c) $W \cap W^\perp = 0$.

<u>Proof</u> (b) $\Leftrightarrow$ (c).

Let $w \in W \cap W^\perp$ and write

$$w = c_1 w_1 + \cdots + c_k w_k$$

Since

$$0 = \langle w_i, w \rangle = \sum_{l=1}^{k} \langle w_i, c_l w_l \rangle$$

$$= \sum_{l=1}^{k} \bar{c}_l \, G(i,l) \quad \text{then}$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \langle w_1, w \rangle \\ \vdots \\ \langle w_k, w \rangle \end{pmatrix} = G \begin{pmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_k \end{pmatrix}$$

So the columns of G are linearly independent if and only if $W \cap W^\perp = 0$.

So G is invertible if and only if $W \cap W^\perp = 0$.

(a) $\Leftrightarrow$ (b) Define

$$w^i = \sum_{l=1}^{k} G^{-1}(i,l) w_l \quad \text{for } i \in \{1, \ldots, k\}.$$

Then

$$\langle w^i, w_j \rangle = \sum_{l=1}^{k} G^{-1}(i,l) \langle w_l, w_j \rangle$$

$$= \sum_{l=1}^{k} G^{-1}(i,l) G(l,j) = \delta_{ij}.$$

Thus the dual basis $\{w^1, \ldots, w^k\}$ exists if and only if G is invertible.