Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form.

Let W be a subspace of V with $k = \dim(V)$.

Let $\{w_1, \ldots, w_k\}$ be a basis of W.

The <u>Gram matrix of $\langle \rangle$ with respect to</u> <u>$\{w_1, \ldots, w_k\}$</u> is $G \in M_k(\mathbb{F})$ given by

$$G(i,j) = \langle w_i, w_j \rangle .$$

The <u>dual basis of W</u> is $\{w^1, \ldots, w^k\}$ such that

$$\langle w^i, w_j \rangle = \delta_{ij} .$$

<u>Example</u> Let $V = \mathbb{R}^3$ with standard dot product.

Let $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$ and basis $\{w_1, w_2\}$ with $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$. The Gram matrix is

$$G = \begin{pmatrix} 3 & 6 \\ 6 & 20 \end{pmatrix} \quad \text{since} \quad \begin{matrix} w_1 \cdot w_1 = 3 & w_1 \cdot w_2 = 6 \\ w_2 \cdot w_1 = 6 & w_2 \cdot w_2 = 20 \end{matrix}$$

The dual basis $\{w^1, w^2\}$ has

$$w^1 = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \\ -\frac{1}{6} \end{pmatrix} \quad \text{and} \quad w^2 = \begin{pmatrix} 0 \\ -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \quad \text{since} \quad \begin{matrix} w^1 \cdot w_1 = 1 & w^1 \cdot w_2 = 0 \\ w^2 \cdot w_1 = 0 & w^2 \cdot w_2 = 1 \end{matrix}$$

and

$$w^1 = \frac{5}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{-1}{4}\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \qquad \text{and} \qquad G^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{-1}{4} \\ -\frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

$$w^2 = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$

Let $\{w^1, \dots w^k\}$ be the dual basis to $\{w_1, \dots, w_k\}$.

The <u>orthogonal projection onto W</u> is the linear transformation

$P_W : V \to V$ given by $P_W(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w^i$

<u>Proposition</u> P_W is the unique linear transf. such that

(1) If $v \in V$ then $P_W(v) \in W$

(2) If $v \in V$ and $w \in W$ then

$$\langle v, w \rangle = \langle P(v), w \rangle$$

<u>Proof</u> To show:

(a) P_W satisfies (1) and (2)

(b) If P and Q satisfy (1) and (2) then $P = Q$.

(a) (1) Assume $v \in V$.

To show: $P_W(v) \in W$.

Since $\{w^1, \dots, w^k\}$ is a basis of W then

$$P_W(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w^i \in W.$$

(a) (2) Assume $v \in V$ and $w \in W$.

To show: $\langle v, w\rangle = \langle P(v), w\rangle$.

Write $w = c_1 w_1 + \cdots + c_k w_k$. Then

$$\langle P_W(v), w\rangle = \left\langle \sum_{i=1}^{k} \langle v, w_i\rangle w_i, \sum_{j=1}^{k} c_j w_j \right\rangle$$

$$= \sum_{j,i=1}^{k} \langle v, w_i\rangle \overline{c_j} \langle w_i, w_j\rangle = \sum_{i=1}^{k} \langle v, c_i w_i\rangle$$

$$= \langle v, w\rangle.$$

So P_W satisfies (1) and (2).

(b) Assume P and Q satisfy (1) and (2).

To show: $P = Q$

To show: If $v \in V$ then $P(v) = Q(v)$.

Assume $v \in V$.

By property (1), $P(v) - Q(v) \in W$.

By property (2), if $w \in W$ then

$$\langle P(v) - Q(v), w\rangle = \langle P(v), w\rangle - \langle Q(v), w\rangle$$

$$= \langle v, w\rangle - \langle v, w\rangle = 0.$$

So $P(v) - Q(v) \in W^\perp$.

So $P(v) - Q(v) \in W \cap W^\perp = \{0\}$.

So $P(v) = Q(v)$ and $P = Q$. //

25.09.2020
GTLA Lecture 4

Orthogonal decomposition

Theorem Let $W \subseteq V$ be a subspace of V such that $\dim(W) \in \mathbb{Z}_{>0}$ and $W \cap W^\perp = 0$.

Then $V = W \oplus W^\perp$.

Proof

(1) If $w \in W$ then $w - P_W(w) \in W$ and $w - P_W(w) \in W^\perp$ since if $w' \in W$ then

$$\langle w - P_W(w), w' \rangle = \langle w, w' \rangle - \langle P_W(w), w' \rangle$$
$$= \langle w, w' \rangle - \langle w, w' \rangle = 0.$$

So $w - P_W(w) \in W \cap W^\perp$.

So $w - P_W(w) = 0$.

So if $w \in W$ then $P_W(w) = w$.

So $\operatorname{im} P_W = W$ and $P_W^2 = P_W$.

Let $P_{W^\perp} = 1 - P_W$.

If $v \in V$ then $P_{W^\perp}(v) = v - P_W(v)$ and $v - P_W(v) \in W^\perp$ since if $w' \in W$ then

$$\langle v - P_W(v), w' \rangle = \langle v, w' \rangle - \langle P_W(v), w' \rangle$$
$$= \langle v, w' \rangle - \langle v, w' \rangle.$$

So $\operatorname{im} P_{W^\perp} = W^\perp$ and

$$(P_{W^\perp})^2 = (1-P_W)(1-P_W) = 1 - 2P_W + P_W^2$$
$$= 1 - 2P_W + P_W = 1 - P_W = P_{W^\perp}.$$

So

$$P_W^2 = P_W, \quad P_{W^\perp}^2 = P_{W^\perp} \text{ and } P_W + P_{W^\perp} = 1.$$

If $v \in V$ then

$$v = 1v = P_W(v) + P_{W^\perp}(v) \in W + W^\perp.$$

Since $W \cap W^\perp = 0$ then

$$V = W \oplus W^\perp. \quad //$$