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GTLA Lecture

Let V be an $\mathbb{F}$-vector space with a sesquilinear form $\langle , \rangle : V \times V \to \mathbb{F}$.

Let $W \subseteq V$ be a subspace and assume $k \in \mathbb{Z}_{>0}$ and $\{w_1, \dots, w_k\}$ is a basis of W.

A <u>dual basis</u> to $\{w_1, \dots, w_k\}$ is $\{w^1, \dots, w^k\}$ in W such that

$$\langle w^i, w_j \rangle = \delta_{ij}.$$

The <u>Gram matrix</u> with respect to $\{w_1, \dots, w_k\}$ is

$$G \in M_k(\mathbb{F}) \text{ given by } G(i,j) = \langle w_i, w_j \rangle.$$

<u>Proposition</u> The following are equivalent:

(a) $\{w^1, \dots, w^k\}$ exists

(b) G is invertible

(c) $W \cap W^\perp = 0$

<u>Proof</u> (b) $\Leftrightarrow$ (c)

Let $w \in W \cap W^\perp$ and write

$$w = c_1 w_1 + \dots + c_k w_k.$$

Since

$$0 = \langle w_i, w \rangle = \sum_{l=1}^{k} \langle w_i, c_l w_l \rangle = \sum_{l=1}^{k} \langle w_i, w_l \rangle \bar{c}_l$$

$$= i\text{th entry of } G \begin{pmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_k \end{pmatrix}$$

then $\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} = \begin{pmatrix}\langle w_1, w\rangle \\ \vdots \\ \langle w_k, w\rangle\end{pmatrix} = G\begin{pmatrix}\bar{c}_1\\ \vdots \\ \bar{c}_k\end{pmatrix}$

So the columns of G are linearly independent if and only if $W \cap W^\perp = 0$.

<u>Our example</u> $V = \mathbb{R}^3$ with standard dot product and

$$W = \mathrm{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix}\right\} \text{ with } w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad w_2 = \begin{pmatrix}2\\0\\4\end{pmatrix}.$$

has

$$G = \begin{pmatrix}3 & 6\\ 6 & 20\end{pmatrix} \text{ and } w^1 = \begin{pmatrix}\frac{1}{3}\\ \frac{5}{6}\\ -\frac{1}{6}\end{pmatrix} \text{ and } w^2 = \begin{pmatrix}0\\ -\frac{1}{4}\\ \frac{1}{4}\end{pmatrix}$$

Assume $\langle\, , \rangle$ is Hermitian and $\mathbb{F}$ has good square roots.
An <u>orthonormal basis</u>, or <u>self dual basis</u>, is a basis of W,

$$\{u_1, \ldots, u_k\} \text{ such that } \langle u_i, u_j\rangle = \delta_{ij}.$$

<u>Gram-Schmidt</u>

Produce $\{u_1, \ldots, u_k\}$ from $\{w_1, \ldots, w_k\}$ one step at a time.

$C = \{w_1, \ldots, w_k\}$, $\quad C_1 = \{b_1, w_2, w_3, \ldots, w_k\}$ with $b_1 = w_1$

$C_2 = \{b_1, b_2, w_3, w_4, \ldots, w_k\}$ with

$$b_2 = w_2 - \frac{\langle w_2, b_1\rangle}{\langle b_1, b_1\rangle} b_1$$

(so that $\langle b_2, b_1\rangle = 0$).

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$C_3 = \{b_1, b_2, b_3, w_4, w_5, \ldots, w_k\}$ with

$$b_3 = w_3 - \frac{\langle w_3, b_1\rangle}{\langle b_1, b_1\rangle} b_1 - \frac{\langle w_3, b_2\rangle}{\langle b_2, b_2\rangle} b_2 \quad \text{so that } \langle b_3, b_2\rangle = 0, \ \langle b_3, b_1\rangle = 0.$$

$\vdots$

$C_k = \{b_1, b_2, \ldots, b_k\}$ with

$$b_k = w_k - \frac{\langle w_k, b_1\rangle}{\langle b_1, b_1\rangle} b_1 - \cdots - \frac{\langle w_k, b_k\rangle}{\langle b_k, b_k\rangle} b_k \quad \text{so that } \langle b_k, b_1\rangle = 0, \ \ldots, \ \langle b_k, b_{k-1}\rangle = 0.$$

$U = \{u_1, u_2, \ldots u_k\}$ given by

$$u_1 = \frac{1}{\sqrt{\langle b_1, b_1\rangle}} b_1, \ \ldots, \ u_k = \frac{1}{\sqrt{\langle b_k, b_k\rangle}} b_k.$$

<u>Example</u> $w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $w_2 = \begin{pmatrix}2\\0\\4\end{pmatrix}$.

$b_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $b_2 = w_2 - \frac{\langle w_2, b_1\rangle}{\langle b_1, b_1\rangle} w_1$

$$= \begin{pmatrix}2\\0\\4\end{pmatrix} - \frac{2+4}{1+1+1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\4\end{pmatrix} - \begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}0\\-2\\2\end{pmatrix}$$

(then $\langle b_1, b_2\rangle = 0$).

Let $u_1 = \begin{pmatrix}\frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\end{pmatrix}$ and $u_2 = \begin{pmatrix}0\\ -\frac{1}{\sqrt2}\\ +\frac{1}{\sqrt2}\end{pmatrix}$

Then $\{u_1, u_2\}$ is orthonormal.

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Orthogonal projection

Let $\mathbb{F}$ be a field. Let V be an $\mathbb{F}$-vector space. Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form on V.

Let $W \subseteq V$ be a subspace of V and assume $\{w_1, \ldots, w_k\}$ is a basis of W and $W \cap W^\perp = 0$.

Let $\{w^1, \ldots, w^k\}$ be the dual basis to $\{w_1, \ldots, w_k\}$.

The <u>orthogonal projection</u> onto W is the linear transformation

$P_W : V \to V$ given by $P_W(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w^i$

<u>Proposition</u> P_W is the unique linear transformation $P: V \to V$ such that

(a) If $v \in V$ then $P(v) \in W$

(b) If $v \in V$ and $w \in W$ then

$$\langle v, w \rangle = \langle P(v), w \rangle.$$

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<u>Proof</u> To show: (a) P_W satisfies (1) and (2)

(b) If P and Q satisfy (1) and (2) then $P=Q$.

(a) Since $\{w^1, \ldots, w^k\}$ is a basis of W then

$$P_W(v) = \sum_{i=1}^{k} \langle v, w_i\rangle w^i \in W.$$

Assume $w \in W$. Then write $w = c_1 w_1 + \cdots + c_k w_k$.

Then

$$\langle P_W(v), w\rangle = \left\langle \sum_{i=1}^{k} \langle v, w_i\rangle w^i, \sum_{j=1}^{k} c_j w_j \right\rangle$$

$$= \sum_{i=1}^{k} \overline{c_i} \langle v, w_i\rangle = \sum_{i=1}^{k} \langle v, c_i w_i\rangle = \langle v, w\rangle.$$

So P_W satisfies (1) and (2).

(b) Assume P and Q satisfy (1) and (2).

To show: If $v \in V$ then $P(v) = Q(v)$.

Assume $v \in V$.

By (1) then $P(v) - Q(v) \in W$.

By (2), if $w \in W$ then

$$\langle P(v) - Q(v), w\rangle = \langle P(v), w\rangle - \langle Q(v), w\rangle$$

$$= \langle v, w\rangle - \langle v, w\rangle = 0.$$

So $P(v) - Q(v) \in W \cap W^\perp$.

So $P(v) - Q(v) = 0$. So $P = Q$. //