Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form.

Let W be a finite dimensional subspace of V with $W \cap W^{\perp} = 0$.

The <u>orthogonal projection onto W</u> is the unique linear transformation $P_W : V \to V$ such that

(1) If $v \in V$ then $P_W(v) \in W$

(2) If $v \in V$ and $w \in W$ then $\langle v, w \rangle = \langle P_W(v), w \rangle$.

<u>Theorem</u> (Orthogonal decomposition)

$$W = W \oplus W^{\perp}$$

<u>Proof</u> (a) If $w \in W$ then $w - P_W(w) \in W$

<u>and</u> $w - P_W(w) \in W^{\perp}$ since

if $w' \in W$ then

$$\langle w - P_W(w), w' \rangle = \langle w, w' \rangle - \langle P_W(w), w' \rangle$$
$$= \langle w, w' \rangle - \langle w, w' \rangle = 0$$

So $w - P_W(w) \in W \cap W^{\perp}$.

So $w - P_W(w) = 0$.

So if $w \in W$ then $P_W(w) = w$.

(b) So $\mathrm{im}\, P_W = W$ and $P_W^2 = P_W$.

(c) Let $P_{W^\perp} = 1 - P_W$

If $v \in V$ then $P_{W^\perp}(v) = v - P_W(v)$ and

$v - P_W(v) \in W^\perp$ since

if $w' \in W$ then

$$\langle v - P_W(v), w' \rangle = \langle v, w' \rangle - \langle P_W(v), w' \rangle$$
$$= \langle v, w' \rangle - \langle v, w' \rangle = 0.$$

So $\mathrm{im}\, P_{W^\perp} \subseteq W^\perp$ and

$$(P_{W^\perp})^2 = (1-P_W)(1-P_W) = 1 - 2P_W + P_W^2 = 1 - P_W = P_{W^\perp}.$$

So $P_W^2 = P_W$, $P_{W^\perp}^2 = P_{W^\perp}$ and $1 = P_W + P_{W^\perp}$

If $v \in V$ then

$$v = 1 \cdot v = P_W(v) + P_{W^\perp}(v) \in W + W^\perp.$$

So $V = W + W^\perp$ and $W \cap W^\perp = 0$.

So $V = W \oplus W^\perp$.

//.

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form.

Let W be a finite dimensional subspace with

$W \cap W^\perp = 0$.

Let $f : W \to W$ be a linear transformation.

The <u>adjoint of f</u> is $f^* : W \to W$ such that

if $x, y \in W$ then $\langle f(x), y\rangle = \langle x, f^*(y)\rangle$

Let $\{w_1, \ldots, w_k\}$ be a basis of W and

let $\{w^1, \ldots, w^k\}$ be the dual basis with respect to $\langle , \rangle$.

If $w = c_1 w_1 + \cdots + c_k w_k$ then $\langle w^i, w\rangle = \overline{c_i}$.

and $w = \overline{\langle w^1, w\rangle} w_1 + \cdots + \overline{\langle w^k, w\rangle} w_k$

So $f^*(y) = \sum_{i=1}^{k} \overline{\langle w^i, f^*(y)\rangle}\, w_i = \sum_{i=1}^{k} \overline{\langle f(w^i), y\rangle}\, w_i$

is a formula for f^* in terms of f.

A linear transformation $f : W \to W$ is

(a) <u>self-adjoint</u> if $f = f^*$

(i.e. if $x, y \in W$ then $\langle f(x), y\rangle = \langle x, f(y)\rangle$)

(d) an <u>isometry</u> if $ff^* = 1$

(i.e. if $x, y \in W$ then $\langle f(x), f(y) \rangle = \langle x, y \rangle$).

(e) <u>normal</u> if $ff^* = f^* f$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$.

Since $e_i \cdot e_j = \delta_{ij}$ then $\{e_1, \ldots, e_n\}$ is orthonormal (self-dual).

Let

$$A \in M_n(\mathbb{C}) \quad \text{and} \quad \begin{aligned} f : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v &\longmapsto Av \end{aligned}$$

Then the matrix of $f^* : \mathbb{C}^n \to \mathbb{C}^n$ with respect to $\{e_1, \ldots, e_n\}$ is A^* given by

$$A^*(i,j) = \overline{A(j,i)} \quad (\text{i.e. } A^* = \bar{A}^t)$$

since

$$\sum_{\ell=1}^{n} A^*(\ell,i) e_\ell = A^* e_i = f^*(e_i)$$

$$= \sum_{\ell=1}^{n} \overline{\langle f(e_\ell), e_i \rangle} e_\ell = \sum_{\ell=1}^{n} \overline{(Ae_\ell \cdot e_i)} e_\ell$$

$$= \sum_{j,\ell=1}^{n} \overline{\langle A(j,\ell) e_j, e_i \rangle} e_\ell$$

$$= \sum_{\ell=1}^{n} \overline{A(i,\ell)} e_\ell.$$

The matrix A is

(1) Hermitian if $A = A^*$

(2) unitary if $AA^* = 1$

(3) normal if $AA^* = A^*A$